CHAPTER 2. Hydrodynamic equations, basic approximations

2.1. Eulerian and Lagrangian description of the flow

Lagrangian flow description looks at coordinates of selected fluid particles \((x(t), y(t), z(t))\) as a function of time. The particle coordinates form in the course of time evolution the trajectories.

In Eulerian flow descriptions we look at the generic variable describing the fluid property per unit mass \(\psi = \psi(x, y, z, t)\) which full differential along the trajectory of Lagrangian fluid particle is

\[
d\psi = \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz .
\]  

(2.1)

Eulerian current components are defined as time derivative of the coordinates \((x(t), y(t), z(t))\) of the passing Lagrangian fluid particles

\[
(u, v, w) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).
\]  

(2.2)

The material time derivative is defined as

\[
\frac{d\psi}{dt} = \left(\frac{\partial \psi}{\partial t}\right)_{local\ time\ derivative} + u\left(\frac{\partial \psi}{\partial x}\right)_{adveective\ derivative} + v\left(\frac{\partial \psi}{\partial y}\right) + w\left(\frac{\partial \psi}{\partial z}\right)
\]  

(2.3)

Eulerian flow description is also characterized by evolving streamlines. At every time instant the flow is directed along the tangent of the streamlines.

2.2. General conservation law

Hydrodynamic (HD) and thermodynamic (TD) equations originate from the conservation laws for mass, 3 momentum components, heat and salt.

General conservation equation of variable \(\psi\) in the fluid volume \(V\) surrounded by a permeable surface \(D\) with a differential element \(d\vec{D} = dD\vec{n}\) is written in a form

\[
\iiint_V \frac{\partial}{\partial t} (\rho \psi) dV = - \oiint_D (\rho \psi) \vec{v} \cdot d\vec{D} + \iiint_V q_\psi dV .
\]  

(2.4)

The second term is converted using the Gauss divergence theorem
\begin{align}
\oint (\rho \psi) \mathbf{v} \cdot d\mathbf{B} &= \iiint \nabla \cdot (\rho \psi) \mathbf{v} \, dV = \iiint \left[ \frac{\partial}{\partial x} (u \rho \psi) + \frac{\partial}{\partial y} (v \rho \psi) + \frac{\partial}{\partial z} (w \rho \psi) \right] \, dV. \tag{2.5}
\end{align}

Omitting the integrals we obtain
\begin{align}
\frac{\partial}{\partial t} (\rho \psi) + \nabla \cdot (\rho \psi \mathbf{v}) &= q_\psi \tag{2.6}
\end{align}
or
\begin{align}
\frac{d}{dt} (\rho \psi) + \rho \psi \nabla \cdot \mathbf{v} &= q_\psi. \tag{2.7}
\end{align}

### 2.3. Mass conservation and continuity

For the mass conservation we take $\psi = 1$ and $q_\psi = 0$, then
\begin{align}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \tag{2.8}
\end{align}
or
\begin{align}
\frac{d \rho}{dt} + \rho \nabla \cdot \mathbf{v} &= 0. \tag{2.9}
\end{align}

For incompressible fluids density is constant along the trajectory
\begin{align}
\frac{d \rho}{dt} &= 0 \tag{2.10}
\end{align}
or
\begin{align}
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho &= 0. \tag{2.11}
\end{align}

This leads to the non-divergence of velocity field
\begin{align}
\nabla \cdot \mathbf{v} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.12}
\end{align}

continuity equation
2.4. Advection-diffusion equation

Advection-diffusion equation of a property \( \psi \) is derived with an assumption that diffusive flux \( v_{m} \nabla \psi \cdot d \bar{D} \) through the permeable surface \( D \) surrounding the volume \( V \) is added to the advective flux \(- (\rho \psi) \bar{v} \cdot d \bar{D} \). Then for incompressible fluid we obtain

\[
\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} + w \frac{\partial \psi}{\partial z} = v_{m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + \bar{P} - \bar{D} \tag{2.13}
\]

where \( \frac{\bar{q}_{\psi}}{\rho} = \bar{P} - \bar{D} \) is production minus destruction of quantity \( \psi \) per unit volume and \( v_{m} \) is molecular diffusion coefficient.

2.5. Turbulent exchange, gradient-dependent fluxes

Turbulent motions are treated by Reynolds decomposition of flow properties into the mean, regular flow component and the irregular pulsation component. Reynolds rules for the property \( \psi \) are written

\[
\psi = \overline{\psi} + \psi', \quad \overline{\psi'} = 0, \quad \overline{\psi} = \bar{\psi}. \tag{2.14}
\]

Substituting the decompositions into the advection-diffusion equation, we obtain after the averaging procedure

\[
\frac{\partial \overline{\psi}}{\partial t} + u \frac{\partial \overline{\psi}}{\partial x} + v \frac{\partial \overline{\psi}}{\partial y} + w \frac{\partial \overline{\psi}}{\partial z} = v_{m} \left( \frac{\partial^2 \overline{\psi}}{\partial x^2} + \frac{\partial^2 \overline{\psi}}{\partial y^2} + \frac{\partial^2 \overline{\psi}}{\partial z^2} \right) + \bar{P} - \bar{D} - \left( \frac{\partial}{\partial x} u' \psi' + \frac{\partial}{\partial y} v' \psi' + \frac{\partial}{\partial z} w' \psi' \right) \tag{2.15}
\]

where additional "turbulent terms" appear due to non-linearity of advection.

Turbulent fluxes are usually parameterised by gradient approach

\[
\overline{u' \psi'} = -\mu_{\psi} \frac{\partial \overline{\psi}}{\partial x}; \quad \overline{v' \psi'} = -\mu_{\psi} \frac{\partial \overline{\psi}}{\partial y}; \quad \overline{w' \psi'} = -\nu_{\psi} \frac{\partial \overline{\psi}}{\partial z}, \tag{2.16}
\]

assuming they act like diffusion or viscosity whereas horizontal and vertical turbulence coefficients \( \mu_{\psi}, \nu_{\psi} \) have different value and generally are not constants. Since turbulent diffusion is much greater than molecular, the latter can be neglected. We will reach the

Advection-diffusion equation of turbulent motions, based on the gradient closure, is written
\[
\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} + w \frac{\partial \psi}{\partial z} = \frac{\partial}{\partial x} \left( \begin{array}{c} \mu_x \\ \mu_y \\ \mu_z \end{array} \right) \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial y} \left( \begin{array}{c} \mu_x \\ \mu_y \\ \mu_z \end{array} \right) \frac{\partial \psi}{\partial y} + \frac{\partial}{\partial z} \left( \begin{array}{c} \mu_x \\ \mu_y \\ \mu_z \end{array} \right) \frac{\partial \psi}{\partial z} + \partial \frac{\partial \psi}{\partial z} + \dot{P} - \dot{D} \quad (2.17)
\]

2.6. Momentum equations

Equations of motion or momentum equations follow the Second Newton Law

\[
\frac{d}{dt} (\rho \vec{v}) = \sum \vec{F} \quad (2.18)
\]

where \( \vec{F} \) is the sum of mass forces and surfacial forces (stresses) acting on a fluid element of unit mass.

In the Newtonian fluid the normal stress (acting across an area in the fluid) is expressed by pressure gradient \(-\nabla p\) and tangential stresses (acting parallel to an area in the fluid) are expressed by viscosity \(\nu \cdot \nabla \vec{v}\) where \(\mu_m\) is the molecular viscosity coefficient.

We will reach then Navier-Stokes equations of motion

\[
\frac{\partial}{\partial t} (\rho \vec{v}) + \vec{v} \cdot \nabla (\rho \vec{v}) = \dot{m} \rho \vec{m} + \nabla p + \nabla \cdot (\mu_m \nabla \vec{v}) \quad (2.19)
\]

For turbulent motions, the friction terms are updated to include turbulent viscosity usually based on the gradient closure scheme.

2.7. System of equations for geophysical fluids

In geophysics the main mass forces are:

- effective gravity force, gravity of Earth acting together with centrifugal force perpendicular to the geopotential surface (undisturbed sea level);
- Coriolis force deflecting motion in the Northern Hemisphere to the right due to Earth rotation;
- tidal forces acting periodically due to gravity forces of Moon and Sun on rotating Earth.

Cartesian coordinate system is used in \(f\) - plane approximation for medium-scale studies. Common axis orientation is east (x), north (y) and up (z) (Fig. 2.1).
Mass forces on a $f$-plane:

- gravity \( \frac{\vec{m}_g}{\rho} = (0,0,-g) \)
- Coriolis force \( \frac{\vec{m}_c}{\rho} = (fv,-fu,0) \)

where

\[ f = 2\Omega \sin \varphi \text{ is Coriolis parameter} \]
\[ \Omega \text{ - rotation speed of Earth} \]
\[ \varphi \text{ - geographical latitude of coordinate origin} \]

**Figure 2.1.** A scheme of $f$-plane.

Equations used in 3D circulation models are:

- **horizontal momentum equations**

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fu &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\mu}{\rho} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\mu}{\rho} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\mu}{\rho} \frac{\partial u}{\partial z} \right) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left( \frac{\mu}{\rho} \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\mu}{\rho} \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\mu}{\rho} \frac{\partial v}{\partial z} \right)
\end{align*}
\] (2.20) (2.21)

- **vertical momentum equation in hydrostatic approximation**

\[
\frac{\partial p}{\partial z} = -g\rho
\] (2.22)

alternative: non-hydrostatic vertical momentum equation

\[
\begin{align*}
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left( \frac{\mu}{\rho} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\mu}{\rho} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\mu}{\rho} \frac{\partial w}{\partial z} \right) + v \frac{\partial w}{\partial z} - g
\end{align*}
\] (2.23)

- **continuity equation**

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\] (2.24)

- **advection-diffusion equation for temperature**
The equations have to follow the boundary conditions:

1) at lateral walls and bottom:
   absence of flow and turbulent flux across the "closed" boundary
   \[ u_n = 0 \quad \frac{\partial T}{\partial n} = 0 \quad \frac{\partial S}{\partial n} = 0 \]  \hfill (2.28)

2) at "open" boundaries: prescribed values or gradients, constituting "remote forcing"

3) forcing at the sea surface
   wind stress \([\text{N/m}^2/\text{s}]\)
   \[ \tau_x = -\rho \, \nu \, \frac{\partial u}{\partial z} \quad \tau_y = -\rho \, \nu \, \frac{\partial v}{\partial z} \]  \hfill (2.29)
   heat flux (temperature flux) and salt flux (units needed to be specified)
   \[ F_Q = -\nu_T \, \frac{\partial T}{\partial z} \quad F_S = -\nu_s \, \frac{\partial S}{\partial z}. \]  \hfill (2.30)

### 2.8. Shallow water equations

When the water depth is much less than horizontal scale of motions \( \delta = \frac{\tilde{H}}{L} \ll 1 \) and water is vertically well-mixed, fluid motion is well described by vertically integrated hydrodynamic equations in \((x, y, t)\) coordinates. We integrate over the whole depth range of water thickness \( h \) from the bottom \( z = -H \) to the surface elevated from the rest by \( \zeta \), \( h = H + \zeta \) and obtain 2D shallow water equations.

Basic variables are:

1) Water level \( \zeta \) that is related to pressure by integrating the hydrostatic equation
   \[ p(x, y, z, t) = p_s(x, y, t) - g\tilde{\rho}(x, y, t)z + g\tilde{\rho}(x, y, t)\zeta(x, y, t) \]  \hfill (2.31)
2) Depth-averaged velocity components and constituents (temperature, salinity, density, substance concentration etc)

\[
\tilde{u} = \frac{1}{h} \int_{-h}^{z} u \, dz \quad \tilde{v} = \frac{1}{h} \int_{-h}^{z} v \, dz \quad \tilde{\psi} = \frac{1}{h} \int_{-h}^{z} \psi \, dz
\]  

(2.32)

2D shallow water equations are:

1) continuity equation

\[
\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial x} (\tilde{u} \tilde{u}^2 + \tilde{h} \tilde{v} \tilde{v}) - f \tilde{h} \tilde{u} = -g h \frac{\partial \tilde{\psi}}{\partial x} \frac{\partial \tilde{\psi}}{\partial y} + \tilde{\mu} \Delta (\tilde{h} \tilde{u}) + F_x - B_x
\]

(2.33)

2) horizontal momentum equations

\[
\frac{\partial}{\partial t} \left( \tilde{h} \tilde{u} \right) + \frac{\partial}{\partial x} \left( \tilde{h} \tilde{u} \tilde{v} + \tilde{h} \tilde{v} \tilde{v} - f \tilde{h} \tilde{v} \right) = -g h \frac{\partial \tilde{u}}{\partial x} \frac{\partial \tilde{\psi}}{\partial y} + \frac{g h^2}{2 \tilde{\rho}} \frac{\partial \tilde{\psi}}{\partial y} + \tilde{\mu} \Delta (\tilde{h} \tilde{v}) + F_y - B_y
\]

(2.34)

3) advection-diffusion equation for scalar (transportable) variable

\[
\frac{\partial}{\partial t} \left( \tilde{h} \tilde{\psi} \right) + \frac{\partial}{\partial x} \left( \tilde{h} \tilde{u} \tilde{\psi} + \tilde{h} \tilde{v} \tilde{\psi} \right) = \frac{\partial}{\partial x} \left( \tilde{h} \tilde{\mu} \frac{\partial \tilde{\psi}}{\partial x} \right) + \frac{\partial}{\partial y} \left( \tilde{h} \tilde{\mu} \frac{\partial \tilde{\psi}}{\partial y} \right) + h (\tilde{P} - \tilde{\mu}) 
\]

(2.35)

4) equation of state

\[
\tilde{\rho} = \rho (\tilde{T}, \tilde{S})
\]

(2.36)

Boundary conditions use isolating conditions at the non-permeable coastline:

free-slip conditions \( \tilde{u}_n = 0 \)  
(normal velocity is set to zero),

\[
\tilde{u} = \tilde{v} = 0
\]

(both velocity components are set to zero, require for accounting the horizontal friction),
\[ \frac{\partial \psi}{\partial n} = 0 \] (absence of normal constituent fluxes).

Forcing (energy input) is done by:

wind stress, usually by quadratic drag formulae from the wind speed \((U_w, V_w)\)

\[ F_x = \frac{\rho_u}{\rho} c_a U_w \sqrt{U_w^2 + V_w^2}, \quad F_y = \frac{\rho_u}{\rho} c_a V_w \sqrt{U_w^2 + V_w^2} \] \hspace{1cm} (2.38)

Tidal forces, if necessary, are introduced by periodic functions depending on geographic coordinates.

Wave stress or wave drift (transformation of wind waves) is important only in very shallow coastal areas.

Remote forcing through the "open" boundaries is treated as:
1) prescribing inflow and constituents of river discharge
2) prescribing exchange with the adjacent sea area (an item of art!!)
   - use larger scale model to get dynamically consistent currents and water level, avoid artificial effects
   - at inflow areas specify the properties of imported water

Energy dissipation is done mainly by the bottom friction.

Quadratic approach for bottom friction is often used in numerical models

\[ B_x = c_b \tilde{u} \sqrt{\tilde{u}^2 + \tilde{v}^2}, \quad B_y = c_b \tilde{v} \sqrt{\tilde{u}^2 + \tilde{v}^2} \] \hspace{1cm} (2.39)

where coefficient \(c_b\) depend on Reynolds number, bottom roughness etc. In engineering applications \(c_b = \frac{g \delta n^2}{h^{7/3}}\) is frequently used, where \(n \approx 10^{-2}\) is Manning bottom friction coefficient and \(\delta\) is unit conversion constant

Linear approach

\[ B_x = rh\tilde{u}, \quad B_y = rh\tilde{v} \] \hspace{1cm} (2.40)

is used sometimes for simplicity of analytical solutions.

Horizontal friction, if accounted in the model, also contributes to the energy dissipation.

Other approximations of the basic equations (two-layer ocean, continuously stratified Boussinesq fluid, quasigestrophy) are later introduced when necessary.