CHAPTER 5. Shallow water theory

In the shallow water the characteristic depth $\tilde{H}$ is much smaller than the horizontal scale of motions $\tilde{L}$

$$\delta = \frac{\tilde{H}}{\tilde{L}} \ll 1.$$  \hspace{1cm} (5.1)

It is considered that the flow does not depend on depth. This is exactly true for barotropic ($\rho = \rho_0 = \text{const}$) non-frictional ($\mu = \nu = 0$) motions. Turbulent viscosity introduces vertical current shear (see drift currents, p. 4.3). However, vertically integrated volume transports are affected by the surface and bottom stresses only, see (4.68) and (4.69).

In this chapter we consider mainly the small-amplitude wave motions. We assume that water level deviation $\xi$ is much smaller than depth $H$

$$\xi \ll H$$  \hspace{1cm} (5.2)

and characteristic velocity $\tilde{U}$ is much smaller than characteristic phase speed of the wave

$$\tilde{U} \ll \frac{\tilde{L}}{\tilde{T}},$$  \hspace{1cm} (5.3)

where $\tilde{T}$ is characteristic wave period.

5.1. Vertically integrated continuity equation

We will introduce the following refinements as compared to the steady Sverdrup regime (p. 4.4):

1) in the non-stationary motion, vertical velocity at the undisturbed sea surface $z = 0$ is kinematically equal to the time derivative of the water level $\frac{\partial \xi}{\partial t}$ (in case of large sea level deviations we have to use the full derivative)

$$w \big|_{z=0} = \frac{\partial \xi}{\partial t}$$  \hspace{1cm} (5.4)

2) in the basin of variable depth $H(x,y)$, a free-slip bottom boundary condition (missing velocity projection normal to the bottom) is

$$\bar{v} \big|_{z=-H} \cdot \nabla H = (u,v,w) \big|_{z=-H} \cdot \left( \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, 1 \right) = 0$$  \hspace{1cm} (5.5)
or

\[ w \big|_{z=-H} = - \left( u \big|_{z=-H} \frac{\partial H}{\partial x} + v \big|_{z=-H} \frac{\partial H}{\partial y} \right). \]  

Doing the vertical integration of the continuity equation (2.24) we obtain

\[
\int_{-H}^{0} \frac{\partial u}{\partial x} \, dz + \int_{-H}^{0} \frac{\partial v}{\partial y} \, dz = - \int_{-H}^{0} \frac{\partial w}{\partial z} \, dz = -\left( w \big|_{z=0} - w \big|_{z=-H} \right) = \\
= -\frac{\partial \xi}{\partial t} - \left( u \big|_{z=-H} \frac{\partial H}{\partial x} + v \big|_{z=-H} \frac{\partial H}{\partial y} \right). \quad (5.7)
\]

Due to the small water level deviations (5.2) we may define the volume transport by integrating the velocities from the bottom to the undisturbed surface

\[ U = \int_{-H}^{0} u \, dz, \quad V = \int_{-H}^{0} v \, dz. \quad (5.8) \]

By differentiating the volume transports, we use the rules for definite integrals with variable bounds

\[
\frac{\partial U}{\partial x} = \frac{\partial}{\partial x} \int_{-H}^{0} u \, dz = \int_{-H}^{0} \frac{\partial u}{\partial x} \, dz + u \big|_{z=-H} \frac{\partial H}{\partial x}, \quad (5.9)
\]

\[
\frac{\partial V}{\partial y} = \frac{\partial}{\partial y} \int_{-H}^{0} v \, dz = \int_{-H}^{0} \frac{\partial v}{\partial y} \, dz + v \big|_{z=-H} \frac{\partial H}{\partial y}. \quad (5.10)
\]

Combining (5.7) with (5.9) and (5.10) we obtain the vertically integrated continuity equation

\[
\frac{\partial \xi}{\partial t} = - \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right). \quad (5.11)
\]

Time derivative of water level depends on the divergence of volume transport. When the volume transport is directed towards the coast then water level will rise. Indeed, if the coast is placed to the right from the water at \( x = L \), then \( U \big|_{L} = 0 \), \( \frac{\partial U}{\partial x} \big|_{L} < 0 \) and \( \frac{\partial \xi}{\partial t} \big|_{L} > 0 \).

**5.2. Vertically integrated momentum equations**

By modifying the horizontal momentum equations (2.20), (2.21) we assume:

1) velocities are small (5.3) and non-linear advection terms \( u \frac{\partial u}{\partial x}, \quad v \frac{\partial u}{\partial y} \ldots \) may be neglected

2) horizontal turbulent viscosity is neglected \( \mu = 0 \)
3) density is constant \( \rho = \rho_0 \)
4) due to hydrostatic approach and constant density, pressure depends on the sea level \( \xi \) and vertical coordinate

\[
p = p_0 - g \rho_0 z + g \rho_0 \xi
\]

(5.12)

from where the horizontal pressure gradients depend on the sea level gradients only

\[
\frac{1}{\rho_0} \frac{\partial p}{\partial x} = g \frac{\partial \xi}{\partial x}, \quad \frac{1}{\rho_0} \frac{\partial p}{\partial y} = g \frac{\partial \xi}{\partial y}.
\]

(5.13)

In the above assumptions we integrate the \( x \)-component momentum equation

\[
\frac{\partial u}{\partial t} - f v = -g \frac{\partial \xi}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + \int_0^z d\zeta,
\]

and similar equation for the \( y \)-component. Following (5.8) we obtain the equations for the volume transport components \([m^2 s^{-2}]\)

\[
\frac{\partial U}{\partial t} - fV = -gH \frac{\partial \xi}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \bigg|_{z=0}^{z=-H},
\]

(5.14)

\[
\frac{\partial V}{\partial t} + fU = -gH \frac{\partial \xi}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} \bigg|_{z=0}^{z=-H},
\]

(5.15)

where viscous stresses appear at the surface

\[
z = 0 : \quad \nu \frac{\partial u}{\partial z} = \tau_x, \quad \nu \frac{\partial v}{\partial z} = \tau_y
\]

(5.16)

and on the bottom

\[
z = -H : \quad \nu \frac{\partial u}{\partial z} = \tau_{xb}, \quad \nu \frac{\partial v}{\partial z} = \tau_{yb}.
\]

(5.17)

Wind stress components \( \tau_x, \tau_y \) are parameterized by a quadratic function of the wind velocity (2.38).

Most problematic item in the shallow water models is the bottom friction parameterization since the near-bottom current shears are not explicitly resolved. In the Stommel circulation problem (p. 4.4) the linear bottom friction formulation
\[ \tau_{xb} = rU \quad , \quad \tau_{yb} = rV \] (5.18)

could be used since the situation was anyhow idealized. It can be shown that in a quasi-steady flow regime the magnitude of the bottom stress is proportional to the magnitude of the volume transport. However, the vectors are deflected depending on the direction of wind and geostrophic flow and on the ratio of the Ekman layers to the whole water column thickness. In non-stationary wave-dominated flows quadratic bottom friction

\[ \tau_{xb} = \frac{c_d U}{H^2} \sqrt{U^2 + V^2} \quad , \quad \tau_{yb} = \frac{c_d V}{H^2} \sqrt{U^2 + V^2} \] (5.19)

gives usually better results. Here the quadratic drag coefficient is taken usually \( c_d \approx 2.5 \cdot 10^{-3} \) but it may also take into account the bottom roughness.

Three-dimensional ocean circulation models based on the full set of hydro-thermodynamic equations (2.20)-(2.30), use vertically integrated momentum equations (5.14) and (5.15) in order to calculate barotropic pressure gradients based on the water level continuity equation (5.11). Bottom friction, effects of non-linear advection terms and baroclinic pressure gradients due to density variations are calculated by the three-dimensional part of the model and their effects can be considered as a correction to the wind stress.

### 5.3. Shallow water equations, numerical solution

#### a) equations

Vertically integrated momentum equations and continuity equation form together the shallow water equations

\[ \frac{\partial U}{\partial t} - fV = -gH \frac{\partial \xi}{\partial x} + \tau_x - \tau_{xb} \quad , \] (5.20)

\[ \frac{\partial V}{\partial t} + fU = -gH \frac{\partial \xi}{\partial y} + \tau_y - \tau_{yb} \quad , \] (5.21)

\[ \frac{\partial \xi}{\partial t} = - \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \quad , \] (5.22)

where three unknowns \( U, V, \xi \) can be solved using the three equations (5.20) – (5.22). In addition to the equations, boundary conditions are needed. In the case of free slip the volume transport may be directed along the coastline \( \Gamma(x, y) \) but normal to the coast component \( U_n \) has to be zero. In the open boundary (river mouth, connection to the other sea area) normal component \( Q(\Gamma_n(x, y), t) \) of the volume transport is prescribed.
At the open boundary sea level or sea level gradient may be prescribed instead of normal volume transport (5.23).

\[ U_n = \begin{cases} 0 & \text{at the rigid boundary} \\ Q(\Gamma(x,y),t) & \text{at the open boundary} \end{cases} \quad \text{(5.23)} \]

\textbf{b) numerical solution}

In a realistic basin with complicated coastline and bottom topography, shallow water equations (5.20)-(5.23) are solved numerically. In a most commonly used finite difference method, temporal and spatial derivatives are written as differences along the finite time and/or grid steps.

Explicit time stepping is most easy to implement. Consider evolution equation for a variable $\phi$ in a form $\frac{\partial \phi}{\partial t} = f(\phi)$, where $f(\cdot)$ is a spatial operator from the modeled variables. Explicit time scheme is written as

\[ \frac{\phi^{n+1} - \phi^n}{\Delta t} = f(\phi^n) \quad \text{(5.24)} \]

where superscript denotes the number of subsequent time layer and $\Delta t$ is the value of discrete time step. This way all the spatial derivatives etc not containing the time derivative are taken from the previous time layer.

For the numerical stability reasons, time discretization of the shallow water equations is quite often done as follows

\[ \frac{U^{n+1} - U^n}{\Delta t} - fV^n = -gH \frac{\partial \zeta^n}{\partial x} + \tau_x - \tau_{xb}^n \quad \text{(5.25)} \]

\[ \frac{V^{n+1} - V^n}{\Delta t} + fU^n = -gH \frac{\partial \zeta^n}{\partial y} + \tau_y - \tau_{yb}^n \quad \text{(5.26)} \]

\[ \frac{\zeta^{n+1} - \zeta^n}{\Delta t} = \left( \frac{\partial U^{n+1}}{\partial x} + \frac{\partial V^{n+1}}{\partial y} \right) \quad \text{(5.27)} \]

where in the continuity equation (5.27) the volume transports at time layer $n+1$ are taken from (5.25), (5.26).

For the spatial discretization of (5.25)-(5.27) the domain is covered by a grid with a step $\Delta x = \Delta y$. With a discrete set of values, the subscript $i$ runs along the $x$-axis and $j$ along the $y$-axis. In the Arakawa C-grid the points of definition of $U$, $V$, $\zeta$ are staggered as shown in Fig. 5.1. Depths are given at $\zeta$-points.
Figure 5.1. Placement of points for volume transport and sea level on the Arakawa C-grid, used for numerical solution of the shallow water equations.

Sea level derivative along $x$ is defined at the $U_{i,j}$-point

$$\frac{\partial \xi}{\partial x} \bigg|_{U_{i,j}} = \frac{\xi_{i,j} - \xi_{i-1,j}}{\Delta x}$$ \hspace{1cm} (5.28)

and derivative along $y$ is defined at the $V_{i,j}$-point

$$\frac{\partial \xi}{\partial y} \bigg|_{V_{i,j}} = \frac{\xi_{i,j} - \xi_{i,j-1}}{\Delta x}$$ \hspace{1cm} (5.29)

We reach the following scheme

$$U^{n+1}_{i,j} = U^n_{i,j} + f\Delta t \tilde{V}^n_{i,j} - \frac{g\Delta t}{2\Delta x} \left( H_{i,j} + H_{i-1,j} \right) \left( \xi^n_{i,j} - \xi^n_{i-1,j} \right) + \left( \tau_x - \tau_x^n \right) \Delta t,$$ \hspace{1cm} (5.30)

$$V^{n+1}_{i,j} = V^n_{i,j} - f\Delta t \tilde{U}^n_{i,j} - \frac{g\Delta t}{2\Delta x} \left( H_{i,j} + H_{i,j-1} \right) \left( \xi^n_{i,j} - \xi^n_{i,j-1} \right) + \left( \tau_y - \tau_y^n \right) \Delta t,$$ \hspace{1cm} (5.31)

$$\xi^{n+1} = \xi^n - \frac{\Delta t}{\Delta x} \left( U^{n+1}_{i+1,j} - U^{n+1}_{i,j} + U^{n+1}_{i,j+1} - V^{n+1}_{i,j} \right),$$ \hspace{1cm} (5.32)

where due to the shift of $U_{i,j}$ and $V_{i,j}$-points, spatially averaged volume transports have to be taken into account in the terms of Coriolis force

$$\tilde{U}_{i,j} = \frac{1}{4} \left( U_{i,j} + U_{i+1,j} + U_{i,j-1} + U_{i+1,j-1} \right),$$ \hspace{1cm} (5.33)
For the numerical stability, time and grid step must follow the relation

$$\frac{\Delta x}{\Delta t} \geq \sqrt{g H_{\text{max}}}.$$  \hspace{1cm} (5.35)

which physical content is: long gravitational wave is not allowed to propagate during one time step further than one spatial grid step.

In the problems of water level and coastal circulation, the numerical model (5.30)-(5.34) quite often gives more than 90% of accuracy as compared to the much more complex three-dimensional models.

5.4. Long gravity waves

a) one-dimensional waves in a narrow channel

Consider frictionless ($\tau_{x}=\tau_{y}=0$) free motions ($\tau_{x}=\tau_{y}=0$) in a long and narrow channel oriented along the $x$-axis. Then cross-channel volume transport is missing $V=0$ and Coriolis force will not appear. The equations (5.20)-(5.22) obtain the form of one-dimensional waves

$$\frac{\partial U}{\partial t} = -gH \frac{\partial \xi}{\partial x},$$     \hspace{1cm} (5.36)

$$\frac{\partial \xi}{\partial t} = \frac{\partial U}{\partial x}. \hspace{1cm} (5.37)$$

Differentiating (5.36) by $t$ and (5.37) by $x$ we obtain hyperbolic differential equation

$$\frac{\partial^2 U}{\partial t^2} - gH \frac{\partial^2 U}{\partial x^2} = 0. \hspace{1cm} (5.38)$$

In case of constant depth search for the wave solution in a form

$$U = U_0 e^{i(kx-\omega t)}.$$  \hspace{1cm} (5.39)

Replacing (5.39) into (5.38) we obtain that (5.38) is satisfied if the wave parameters follow

$$\omega^2 - gHk^2 = 0.$$ \hspace{1cm} (5.40)

From the condition (5.40) we get the dispersion relation for one-dimensional long gravity waves
\( \omega = \sqrt{gH} \frac{k}{c}, \quad (5.41) \)

that relates frequency \( \omega \) to the wave number \( k \). Ratio of frequency and wave number \( c = \frac{\omega}{k} \) determines the phase speed.

\( c = \sqrt{gH} \quad (5.42) \)

Long one-dimensional gravity waves are non-dispersive, i.e. the phase speed does not depend on the wave number(s).

If we add (variable) wind stress to the equations (5.36)-(5.37) then we obtain forced waves. Here resonant forcing appears if the wind pattern moves with a phase speed of waves \( c = \sqrt{gH} \). Resonant forcing is often a precondition of creating highest storm surges. Taking the mean depth of the Gulf of Finland 60 m, we obtain floods in St. Petersburg when the cyclone is moving with a speed \( \sqrt{9.8 \times 60} = 24.2 \) m/s.

Consider next the case of narrow channel of a certain length \( L \). At the ends of the channel the volume transport has to vanish

\( U(0,t) = U(L,t) = 0 \quad . \quad (5.43) \)

Search for the solution of (5.38) in a form

\( U = U_0 \cos \omega t \sin k_n x \quad . \quad (5.44) \)

Boundary conditions (5.43) can be satisfied at discrete set of wave numbers

\( k_n = \frac{n\pi}{L}, \quad n = 1, 2... \quad (5.45) \)

The solution (5.44), (5.45) represents the standing waves in the channel. These eigenoscillations are called seisches. Oscillations corresponding to the different discrete wave numbers are called modes (Fig. 5.2).

![Figure 5.2. First, second and third modes of eigenoscillations of a narrow channel: volume transport (left) and water level (right).](image)
b) two-dimensional waves

Coriolis force may become important for two-dimensional waves. Assume constant depth \( H = \text{const} \). Search for the periodic solution of (5.20)-(5.22) in a form

\[
U = \tilde{U}(x, y)e^{i\omega x}, \quad V = \tilde{V}(x, y)e^{i\omega y}, \quad \xi = \tilde{\xi}(x, y)e^{i\omega x}.
\]  

(5.46)

For the spatial parts of (5.46) we obtain the equations

\[
i\omega \tilde{U} - f\tilde{V} = -gH \frac{\partial \tilde{\xi}}{\partial x},
\]

(5.47)

\[
i\omega \tilde{V} + f\tilde{U} = -gH \frac{\partial \tilde{\xi}}{\partial y},
\]

(5.48)

\[
i\omega \tilde{\xi} = \left( \frac{\partial \tilde{U}}{\partial x} + \frac{\partial \tilde{V}}{\partial y} \right).
\]

(5.49)

Solve for the spatial parts of volume transports from (5.47), (5.48)

\[
\tilde{U} = \frac{1}{\omega^2 - f^2} \left( i\omega \frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right)gH\tilde{\xi}.
\]

(5.50)

\[
\tilde{V} = \frac{1}{\omega^2 - f^2} \left( -f \frac{\partial}{\partial x} + i\omega \frac{\partial}{\partial y} \right)gH\tilde{\xi}.
\]

(5.51)

Substituting (5.50), (5.51) into the continuity equation (5.49) we obtain

\[
\frac{\partial^2 \tilde{\xi}}{\partial x^2} + \frac{\partial^2 \tilde{\xi}}{\partial y^2} + \frac{\omega^2 - f^2}{gH} \tilde{\xi} = 0.
\]

(5.52)

In case of periodic wave solution \( \tilde{\xi} \) has the form

\[
\tilde{\xi} = \xi_0 \exp i (kx + ly).
\]

(5.53)

Replacing this into (5.52) we obtain dispersion relation of long gravity waves on a rotating frame in the \( f \)-plane approximation

\[
\omega^2 = f^2 + gH(k^2 + l^2).
\]

(5.54)

When both the wave numbers \( k, l \) are real, i.e. the wave profile is periodic in both the \( x \) and \( y \) directions, then frequency \( \omega \) is higher than the inertial frequency \( \omega > f \). Consequently, periodic (in both directions) gravity waves can not have a longer period than the inertial period \( T < T_f \approx 14h \).
In case of longer periods $k^2 + l^2 < 0$ must take place or at least one of the wave numbers have to be imaginary. In this direction the wave profile is exponential. It is possible only near the coast whereas the wave amplitude is maximal at the coast and decays exponentially seaward. Waves with an exponential profile are called Kelvin waves. Kelvin waves appear for example at Baltic Sea seisches and also at diurnal tides. For the seisches of small basins (like the Gulf of Riga) rotational effects do not have remarkable effect.

5.5. Water level equation

When investigating the wave properties apparent in the shallow water equations at different coastline and bottom geometry then it is useful to transform the equations into one equation. Frictionless ($\tau_x = \tau_y = \tau_{sb} = \tau_{yb} = 0$) small amplitude motions are described [see (5.20)-(5.22)] as

$$\frac{\partial U}{\partial t} - fV = -gH \frac{\partial \xi}{\partial x} \ , \quad (5.55)$$

$$\frac{\partial V}{\partial t} + fU = -gH \frac{\partial \xi}{\partial y} \ , \quad (5.56)$$

$$\frac{\partial \xi}{\partial t} = -\left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \ . \quad (5.57)$$

Let us differentiate (5.57) by time

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial}{\partial x} \frac{\partial U}{\partial t} + \frac{\partial}{\partial y} \frac{\partial V}{\partial t} = 0 \ . \quad (5.58)$$

Replacing $\frac{\partial U}{\partial t}$, $\frac{\partial V}{\partial t}$ in (5.58) from (5.55), (5.56) we obtain

$$\frac{\partial^2 \xi}{\partial t^2} - g \left( \frac{\partial}{\partial x} H \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial y} H \frac{\partial \xi}{\partial y} \right) + f \left( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) = 0 \ . \quad (5.59)$$

For eliminating $U, V$ we take once more the time derivative from (5.59)

$$\frac{\partial}{\partial t} \left[ \frac{\partial^2 \xi}{\partial t^2} - g \left( \frac{\partial}{\partial x} H \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial y} H \frac{\partial \xi}{\partial y} \right) + f \left( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) \right] = 0 \ . \quad (5.60)$$

Let us transform the last term of (5.60), replacing $\frac{\partial U}{\partial t}$, $\frac{\partial V}{\partial t}$ from (5.55), (5.56) and taking into account $f = \text{const}$
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\[
f \left( \frac{\partial}{\partial x} \frac{\partial V}{\partial t} - \frac{\partial}{\partial y} \frac{\partial U}{\partial t} \right) = -f^2 \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) - g f \left( \frac{\partial}{\partial x} H \frac{\partial \xi}{\partial y} - \frac{\partial}{\partial y} H \frac{\partial \xi}{\partial x} \right) = 0 \tag{5.61}
\]

\[
= f^2 \frac{\partial^2 \xi}{\partial t^2} - g f J(H, \xi)
\]

where

\[
J(a,b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y} \tag{5.62}
\]

is the Jacobi operator.

Taking into account (5.61), the equation (5.60) is rewritten

\[
\frac{\partial}{\partial t} \left[ \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \xi - g \left( \frac{\partial}{\partial x} H \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y} H \frac{\partial \xi}{\partial x} \right) \right] - g f \left( \frac{\partial H}{\partial x} \frac{\partial \xi}{\partial y} - \frac{\partial H}{\partial y} \frac{\partial \xi}{\partial x} \right) = 0 \tag{5.63}
\]

that is equivalent to the initial equations (5.55)-(5.57).

Consider a boundary condition for (5.63) at wall, which is perpendicular to the \(x\)-axis. In that case \(U = 0\) and \(\frac{\partial U}{\partial t} = 0\). From (5.55), (5.56) we obtain

\[
fV = -gH \frac{\partial \xi}{\partial x}, \quad \frac{\partial V}{\partial t} = -gH \frac{\partial \xi}{\partial y}
\]

or

\[
U = 0 \implies \frac{\partial^2 \xi}{\partial x \partial t} + f \frac{\partial \xi}{\partial y} = 0 \tag{5.64}
\]

Analogically the boundary condition is obtained for the wall perpendicular to the \(y\)-axis

\[
V = 0 \implies \frac{\partial^2 \xi}{\partial y \partial t} - f \frac{\partial \xi}{\partial x} = 0 \tag{5.65}
\]

5.6. General wave solution in a channel of constant depth

With constant depth \(H = \text{const}\) the equation (5.63) is simplified

\[
\frac{\partial}{\partial t} \left[ \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \xi - gH \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) \right] = 0 \tag{5.66}
\]

The highest order of derivatives is two and (5.66) is further reduced to the hyperbolic wave equation

\[
\left( \frac{\partial^2}{\partial t^2} + f^2 \right) \xi - gH \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) = 0 \tag{5.67}
\]
Consider a channel along \(x\)-axis with a width \(L\). Boundary conditions at the channel walls

\[ V = 0 \text{ at } y = 0, L \quad (5.68) \]

are transformed according to (5.65) into a form

\[ \frac{\partial^2 \xi}{\partial y \partial t} - f \frac{\partial \xi}{\partial x} = 0 \text{ at } y = 0, L \quad . \quad (5.69) \]

Search for the solutions periodic by \(x\) and \(t\)

\[ \xi = \text{Re} \bar{\xi}(y) e^{i(k_x x - \omega t)} \quad (5.70) \]

where \(\bar{\xi}(y)\) is complex wave amplitude across the channel, \(k\) is along-channel wave number and \(\omega\) is wave frequency.

Replacing (5.70) into the equation (5.67) and boundary conditions (5.69), we obtain an eigenvalue problem for \(\bar{\xi}\)

\[ \frac{d^2 \bar{\xi}}{dy^2} + \left( \frac{\omega^2 - f^2}{gH} - k^2 \right) \bar{\xi} = 0 \quad , \quad (5.71) \]

\[ \frac{d\bar{\xi}}{dy} + f \frac{k}{\omega} \bar{\xi} = 0 \text{ at } y = 0, L \quad , \quad (5.72) \]

which general solution is

\[ \bar{\xi} = A \sin ly + B \cos ly \quad , \quad (5.73) \]

whereas by (5.71)

\[ l^2 = \frac{\omega^2 - f^2}{gH} - k^2 . \quad (5.74) \]

Here \(l\) is the cross-channel wave number (\(y\)-component of the wave vector \(\vec{k} = (k, l)\)). Substituting general solution (5.73) into the boundary conditions (5.72), we get a homogeneous system of linear equations with respect to \(A, B\)

\[ lA + \frac{fk}{\omega} B = 0 \quad , \quad (5.75) \]

\[ A\left( l \cos lL + f \frac{k}{\omega} \sin lL \right) + B\left( f \frac{k}{\omega} \cos lL - l \sin lL \right) = 0 \quad . \quad (5.76) \]

The above system has non-trivial solutions only then if the determinant is zero.
\[
(\omega^2 - f^2)(\omega^2 - gHk^2)\sin lL = 0 .
\] (5.77)

By the condition (5.77) the eigenvalue problem (5.71), (5.72) has three types of solutions, whereas the case \(\omega^2 - f^2 = 0\) or \(\omega = \pm f\) is spurious.

### 5.7. Poincare waves

Consider the waves corresponding to the case

\[\sin lL = 0 \quad (5.78)\]

in the condition (5.77) of the eigenvalue problem (5.71), (5.72).

Condition (5.78) is satisfied if cross-channel wave number \(l\) takes discrete values

\[l_n = \frac{n\pi}{L} \quad n = 1, 2, .... \quad (5.79)\]

Seeing the definition (5.74) of \(l\) based on the equation (5.67) and wave parameters (5.70), we obtain

\[l_n^2 = \frac{\omega^2 - f^2}{gH} - k^2 = \frac{n^2\pi^2}{L^2} ,
\] (5.80)

where frequency is found as

\[\omega = \omega_n = \pm \sqrt{f^2 + gH\left(k^2 + \frac{n^2\pi^2}{L^2}\right)} ,
\] (5.81)

or

\[\omega_n = \pm \sqrt{f^2 + gH\left(k^2 + l_n^2\right)} .
\] (5.82)

Dispersion relation of Poincare waves (5.82) coincides with that of the two-dimensional waves in the unbounded basin (5.54), only the cross-channel wave numbers obtain only discrete values due to the boundary conditions. Frequency of Poincare waves is always higher than the inertial frequency \(\omega_n > f\) and the frequency of one-dimensional cross-channel eigenoscillations \(\omega_n > \sqrt{gH}l_n\), or the period is shorter than the inertial period

\[T_n = \frac{2\pi}{\omega_n} < T_f = \frac{2\pi}{f} \approx 14 h\]

and the period of cross-channel eigenoscillations

\[T_n < T_L = \frac{2L}{n\sqrt{gH}} .
\]
Poincare waves can propagate both in positive and negative directions of $x$-axis. Phase speed $c_n = \frac{\omega_n}{k}$ is determined from the dispersion relation (5.82) as

$$c_n = \pm \sqrt{gH + \frac{f^2 + gH l_n^2}{k^2}}.$$  \hspace{1cm} (5.83)

See that Poincare waves are dispersive. Remind that non-rotational gravity waves have non-dispersive phase speed $c = \sqrt{gH}$.

Spatial pattern of Poincare waves is derived from (5.73) where coefficients $A, B$ in $\xi_n(y) = A\sin l_n y + B\cos l_n y$ must satisfy boundary conditions (5.72) pre-assuming discrete wave numbers (5.79). The traveling wave is

$$\xi = \xi_0 \left( \cos \frac{n\pi y}{L} - \frac{L}{n\pi \omega} \frac{f k}{\sin \frac{n\pi y}{L}} \right) \cos(kx - \omega t)$$  \hspace{1cm} (5.84)

or

$$\xi = \xi_0 \left( \cos l_n y - \frac{f k}{\omega l_n} \sin l_n y \right) \cos(kx - \omega t) \hspace{1cm} (5.85)$$

Cross-channel water level pattern of Poincare waves is asymmetric and depends on the direction of wave propagation. With $\omega > 0$ the wave propagates to the right and higher water levels appear in the upper half of the channel. With $\omega < 0$ the situation is opposite (Fig. 5.3).

**Figure 5.3.** Water level distribution in a channel due to Poincare waves propagating to the right (above) and to the left (below).
Superposition of two Poincare waves of equal amplitude, traveling in opposite directions, yields standing Poincare wave

\[ \xi = \xi_0 \cos l_n \cos kx \cos \omega t \quad (5.86) \]

where the water level distribution is symmetric and the nodal lines (lines of zero water level) are fixed.

It is useful to scale the wave frequency by inertial frequency and rewrite the dispersion relation (5.81) in a non-dimensional form

\[ \frac{\omega_n}{f} = \sqrt[4]{\frac{R_d}{\lambda_x} + \left( \frac{n R_d}{2L} \right)^2} \quad (5.87) \]

where \( \lambda_x = \frac{2\pi}{k} \) is wavelength [m] and the barotropic (external) Rossby deformation radius [m] is

\[ R_d = \frac{\sqrt{gH}}{f} \quad (5.88) \]

Period of Poincare waves is determined from (5.87) as

\[ \frac{T_n}{T_f} = \left| \frac{f}{\omega_n} \right| \quad \text{At large wavelengths} \lambda_x \to \infty \text{ the period of Poincare waves asymptotically approaches} \quad T_\infty = T_f \frac{1}{\sqrt[4]{1 + 4\pi^2 \left( \frac{n R_d}{2L} \right)^2}} \quad (5.90) \]

By increasing the channel width \( L \to \infty \) the period approaches to the inertial period \( T_\infty \to T_f \). At small wavelengths \( \lambda_x \ll \frac{2L}{n} \) the wave period is proportional to the wavelength \( T = \frac{\lambda_x}{2\pi R_d} \) (Fig. 5.4).

Using the wavelength \( \lambda_x = \frac{2\pi}{k} \) along the x-axis and the relation (5.79) \( l_n = \frac{n \pi}{L} \), we obtain the ratio of the phase speeds of the Poincare waves (5.83) and the one-dimensional waves \( \sqrt{gH} \) in a form

\[ \frac{c_n}{\sqrt{gH}} = \sqrt[4]{1 + \left( \frac{\lambda_x}{R_d} \right)^2 \left[ \frac{1}{4\pi^2} + \left( \frac{n R_d}{2L} \right)^2 \right]} \quad (5.89) \]
Non-dimensional periods \( \frac{T_n}{T_f} \) and phase speeds \( \frac{c_n}{\sqrt{gH}} \) as the functions of non-dimensional wave length \( \frac{\lambda_x}{R_d} \) and channel width \( \frac{2L}{nR_d} \) are shown in Fig. 5.4. With \( H = 100 \) m and \( f = 1.25 \times 10^{-4} \) s\(^{-1}\) we obtain \( \sqrt{gH} = 31.3 \) m/s (phase speed of one-dimensional waves) and \( R_d = \frac{\sqrt{gH}}{f} = 250 \) km (barotropic Rossby deformation radius). Considering typical width of the Baltic Sea \( L = 250 \) km, we get the non-dimensional width for the first mode \( \frac{2L}{R_d} = 2 \) and for the second mode \( \frac{2L}{2R_d} = 1 \).

Figure 5.4. Non-dimensional periods (left) and phase speeds (right) of Poincare waves as a function of non-dimensional along-channel wave length in case of different non-dimensional channel width.

5.8. Kelvin waves

Kelvin waves are obtained from the eigenvalue problem (5.71), (5.72) considering the case

\[ \omega = \pm \sqrt{gH} k \]  

(5.90)

from the condition (5.77). The phase speed \( c = \frac{\omega}{k} = \pm \sqrt{gH} \) does not depend on the wave number and therefore the Kelvin waves are non-dispersive.

From (5.74) we obtain

\[ l^2 = \frac{f^2}{gH} \]  

(5.91)

which means that the cross-channel wave number is imaginary.
\[ l = \pm i \frac{f}{\sqrt{gH}} \] (5.92)

and the cross-channel structure (5.73) is exponential

\[ \bar{\xi} = A \exp\left(-\frac{fy}{\sqrt{gH}}\right) + B \exp\left(\frac{fy}{\sqrt{gH}}\right) \] (5.93)

For the wave traveling in positive direction of the \( x \)-axis \( (\omega > 0) \), the boundary condition (5.72) can be satisfied only then if the water level amplitude decreases with \( y \)-axis

\[ \xi_+ = \xi_0 \exp\left(-\frac{fy}{\sqrt{gH}}\right) \cos (kx - \omega t) \] (5.94)

where \( \omega = |\omega| \). Kelvin wave traveling in the opposite direction can exist only then if the water level amplitude is maximal at the wall \( y = L \)

\[ \xi_- = \xi_0 \exp\left(\frac{f(y-L)}{\sqrt{gH}}\right) \cos (kx + \omega t) \] (5.95)

This way direction of cross-channel amplitude decay depends on the direction of wave propagation. Looking in the direction of Kelvin wave propagation, the amplitude increases to the right in the Northern Hemisphere (Fig. 5.5).

![Figure 5.5. Water level deviations in the Kelvin wave traveling in the direction \( \vec{c} \).](image-url)
The scale for the amplitude decay in a cross-channel direction is \( R_d = \frac{\sqrt{gH}}{f} \) that is also known as the barotropic (external) Rossby deformation radius. For example at \( H = 100 \) m, the phase speed is \( c = \sqrt{gH} = 31.3 \) m/s and the e-fold amplitude decay occurs in a distance \( R_d = 250 \) km. In the shallow water body with \( H = 10 \) m (for example Lake of Peipsi) the phase speed of Kelvin wave is \( c = 10 \) m/s and the e-fold decay distance is \( R_d = 80 \) km.

Within the Kelvin wave, the cross-channel volume transport component is exactly zero

\[
V = 0 \tag{5.96}
\]

everywhere in the channel. Since the cross-channel variation of properties is monotonic then the boundary conditions (5.68) cannot be otherwise satisfied.

Due to (5.96) the shallow water equations describing Kelvin waves are

\[
\frac{\partial U}{\partial t} = -gH \frac{\partial \xi}{\partial x}, \tag{5.97}
\]

\[
\frac{\partial \xi}{\partial t} = -\frac{\partial U}{\partial x}, \tag{5.98}
\]

\[
U = -\frac{gH}{f} \frac{\partial \xi}{\partial y}. \tag{5.99}
\]

As evident from (5.99) the along-channel volume transport is in a geostrophic balance with the cross-channel sea level gradient.

### 5.9. Amphidromic systems of Kelvin waves

Superposition of two Kelvin waves of equal amplitude propagating in opposite directions is [see (5.94) and (5.95)]

\[
\xi = \xi_+ + \xi_- = \xi_0 \left[ \exp \left( -\frac{y}{R_d} \right) + \exp \left( \frac{y - L}{R_d} \right) \cos kx \right] \cos \omega t + \\
+ \xi_0 \left[ \exp \left( -\frac{y}{R_d} \right) - \exp \left( \frac{y - L}{R_d} \right) \sin kx \right] \sin \omega t, \tag{5.100}
\]

where \( R_d \) is the deformation radius (5.88). Expression (5.100) represents the standing Kelvin wave which general form is

\[
\xi = A_1(x, y) \cos \omega t + A_2(x, y) \sin \omega t = \sqrt{A_1^2(x, y) + A_2^2(x, y)} \sin \left( \omega t - \gamma(x, y) + \frac{\pi}{2} \right), \tag{5.101}
\]
where the initial phase dependent on $x, y$ is

$$\gamma(x, y) = \arctan \left( \frac{A_2(x, y)}{A_1(x, y)} \right). \tag{5.102}$$

Expression (5.101) means that the amplitude of oscillations at every point is given by

$$A(x, y) = \sqrt{A_1^2(x, y) + A_2^2(x, y)} \tag{5.103}$$

and the co-range lines are determined by $A(x, y) = \text{const}$. In the case when the amplitude (5.103) is permanently zero $A(x, y) = 0$ at some isolated points, the waves form amphidromic systems and the points of zero amplitude are nodal points of the amphidromic system. Note that initial phase (5.102) cannot be determined in the nodal points. Outside the nodal points the phase of oscillations is different depending on the initial phase (5.102). Co-tidal lines are determined by the equation

$$\hat{\omega}t = \gamma(x, y) \quad \text{or} \quad \tan \hat{\omega}t = \frac{A_2(x, y)}{A_1(x, y)}. \tag{5.104}$$

Co-tidal lines converge in the nodal points. The lines are usually marked by angle $0^\circ \text{ to } 360^\circ$ or time $t = 0 \pm \frac{2\pi}{\omega}$.

Amphidromic systems may appear at different waves. In the oceanography they are mainly related to the Kelvin waves arising from the tidal forces and from the excitation of seiches by the changing winds. In view of (5.100) and (5.101), standing Kelvin waves have specific amplitude distribution in the channel

$$A_1 = \xi_0 \exp \left( -\frac{L}{2R_d} \right) \cosh \left( \frac{y - y_0}{R_d} \right) \cos kx, \quad A_2 = \xi_0 \exp \left( -\frac{L}{2R_d} \right) \sinh \left( \frac{y - y_0}{R_d} \right) \sin kx,$$

where nodal points are $y_0 = \frac{L}{2}$, $x_0 = \frac{n\pi}{k}$. Using the expansion into Taylor series

$$\sinh \left( \frac{y - y_0}{R_d} \right) \approx \frac{y - y_0}{R_d} \quad \text{and} \quad \cosh \left( \frac{y - y_0}{R_d} \right) \approx 1 \quad \text{if} \quad \left| \frac{y - y_0}{R_d} \right| << 1,$$

the equation for co-tidal lines (5.104) is transformed as

$$\tan \hat{\omega}t \approx \frac{y - y_0}{R_d} \tan kx. \tag{5.105}$$

Since $\tan kx \approx kx$ at $|kx| << 1$, the co-tidal lines become strait lines near the nodal point

$$y \approx kR_d x \tan \hat{\omega}t + y_0, \tag{5.106}$$
where tangent of the line \( kR_j \tan \dot{\omega}_t \) increases with time \( t \). Therefore in the Kelvin waves, the oscillation phase rotates in the Northern Hemisphere counterclockwise around the nodal points (Fig. 5.6a). Amphidromic systems of Kelvin waves appear also in the closed and semi-enclosed basins (Fig. 5.6b).

![Figure 5.6. Amphidromic systems: a) Kelvin waves in an infinitely long channel, b) M\(_2\) tides in the North Sea.]

### 5.10. Topographic Rossby waves

Consider a channel with sloping bottom

\[
H = H_0 \left(1 - \frac{\alpha y}{L} \right),
\]

(5.107)

where the bottom slope \( \alpha \ll 1 \) is small. We can take \( H \approx H_0 \) in the equation of water level (5.63), except for the derivatives where we obtain \( \frac{\partial H_0}{\partial y} = -H_0 \frac{\alpha}{L} \). The water level equation takes the form

\[
\frac{\partial}{\partial t} \left[ \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \xi - gH_0 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{\alpha gH_0}{L} \frac{\partial \xi}{\partial y} \right] - \frac{\alpha gH_0 f}{L} \frac{\partial \xi}{\partial x} = 0.
\]

(5.108)
Search for the wave solution as earlier (5.70)

\[ \bar{\xi} = \text{Re} \frac{\bar{\xi}(y)}{e^{i(kx - \omega t)}} \]  

(5.109)

The cross-channel structure \( \bar{\xi}(y) \) must satisfy

\[
-i \omega \left[ (-\omega^2 + f^2) \bar{\xi} + gH_0 k^2 \bar{\xi} - gH_0 \frac{d^2 \bar{\xi}}{dy^2} + \frac{\alpha gH_0}{L} \frac{d \bar{\xi}}{dy} \right] - i \frac{\alpha gH_0 f k}{L} \bar{\xi} = 0
\]

(5.110)

or after simplifications

\[
\frac{d^2 \bar{\xi}}{dy^2} - \frac{\alpha}{L} \frac{d \bar{\xi}}{dy} + \left( \frac{\omega^2 - f^2}{gH_0} - k^2 - \frac{\alpha f k}{L \omega} \right) \bar{\xi} = 0
\]

(5.111)

Boundary conditions are the same as earlier (5.72)

\[
\frac{d \bar{\xi}}{dy} + \frac{f k}{\omega} \bar{\xi} = 0 , \quad y = 0, L
\]

(5.112)

General solution of the equation (5.111) has the form

\[
\bar{\xi} = e^{\frac{\alpha y}{L}} (A \sin ly + B \cos ly)
\]

(5.113)

where

\[
l^2 = \frac{\omega^2 - f^2}{gH_0} - k^2 - \frac{\alpha f k}{L \omega} - \frac{\alpha^2}{4L^2}
\]

(5.114)

Boundary conditions (5.112) yield the eigenvalue problem

\[
\left( \omega^2 - f^2 \right) \left( \omega^2 - gH_0 k^2 \right) \sin lL = 0
\]

(5.115)

that is similar to (5.77) in the case of channel of a constant depth.

We will see that Poincare and Kelvin waves remain nearly unchanged at small bottom slopes, only a slowly changing multiplier \( e^{\frac{\alpha y}{L}} \) appears in the wave amplitude.

From the condition

\[
\sin lL = 0
\]

(5.116)

of the problem (5.115) we obtain discrete cross-channel modes
\[ l_n = \frac{n \pi}{L}. \]  \hspace{2cm} (5.117)

From (5.114) we obtain cubic equation for the frequency

\[ \omega^2 - \frac{\alpha f k g H_0}{L \omega} - g H_0 \left( k^2 + \frac{n^2 \pi^2}{L^2} + \frac{f^2}{g H_0} + \frac{\alpha^2}{4L^2} \right) = 0 . \]  \hspace{2cm} (5.118)

In case of low-frequency waves

\[ \omega^2 \ll \left| \frac{\alpha f k g H_0}{L \omega} \right|, \]  \hspace{2cm} (5.119)

using also condition for small bottom slope \( \frac{\alpha^2}{4L^2} \ll k^2 + \frac{n^2 \pi^2}{L^2} \), we obtain the dispersion relation for the (barotropic) topographic Rossby waves

\[ \omega = \frac{\alpha f}{L} \frac{k}{k^2 + \frac{n^2 \pi^2}{L^2} + \frac{f^2}{g H_0}} \quad n = 1, 2, \ldots \]  \hspace{2cm} (5.120)

Frequency of topographic wave is maximal (period is minimal) at the scales of Rossby deformation radius

\[ k = k_n = \sqrt{\frac{n^2 \pi^2}{L^2} + \frac{f^2}{g H_0}} \quad \text{or} \quad k_n = \sqrt{\frac{n^2 \pi^2}{L^2} + \frac{1}{R_d^2}} \]  \hspace{2cm} (5.121)

Topographic Rossby waves can propagate in one direction only: the shallower water must remain to the right from the direction of wave propagation. Topographic waves are quite slow and the phase speeds are usually less than 1 m/s. For the small bottom slopes, the frequency of topographic waves is always less than the inertial frequency \( f \).

All the barotropic waves considered above in p. 5.7-5.10 are summarized in Fig. 5.7 adopted from J. Pedlosky ("Geophysical Fluid Dynamics", 1979).
Figure 5.7. A scheme of dispersion relations of the barotropic waves appearing in the infinitely long channel: non-modal Kelvin waves propagating in both directions, cross-channel modes of Poincare waves propagating in both directions, cross-channel modes of topographic Rossby waves propagating with shallower water on the right (a figure by J. Pedlosky).