CHAPTER 6. Long waves in a two-layer ocean

6.1. Two-layer shallow water equations

Stratified fluid can be often considered to consist from nearly homogeneous layers of slightly different density. The interface between the layers – pycnocline – can be found between the upper mixed layer and the deeper more cold layers (thermocline) or between the layers of different salinity (halocline in the marginal seas like the Baltic Sea, river plumes in the regions of freshwater influence).

Layer models assume constant densities and velocities in the fluid layers, performing a jump while passing through the interface from one layer to another. At the same time, change of fluid pressure is continuous. Stratification dominated by one pycnocline allows us to use a two-layer approach for the hydrodynamic equations. In the layers with mean thickness $h_1$, $h_2$ the density is assumed constant $\rho_1 = \text{const}$, $\rho_2 = \text{const}$ whereas $\rho_2 > \rho_1$. Deviations of the free surface $\xi$ and the interface between the layers $\eta$ are usually considered small (Fig. 6.1).

![Diagram of a two-layer fluid](image)

Figure 6.1. Schematic presentation of a two-layer fluid.

a) continuity equations

Volume transports in the both layers are calculated from the actual velocities $u_1$, $u_2$, $v_1$, $v_2$ as

$$
U_1 = \int_{-(h+\eta)}^{\xi} u_1 dz, \quad U_2 = \int_{-H}^{-(h+\eta)} u_2 dz, \quad U = U_1 + U_2,
$$

$$
V_1 = \int_{-(h+\eta)}^{\xi} v_1 dz, \quad V_2 = \int_{-H}^{-(h+\eta)} v_2 dz, \quad V = V_1 + V_2.
$$

(6.1)
Assuming vertically constant velocities in the both layers, the volume transports are defined within small deviations of $\xi$ and $\eta$ as

$$
\begin{align*}
U_1 &= u_1 h_1, & U_2 &= u_2 h_2, & U &= U_1 + U_2, \\
V_1 &= v_1 h_1, & V_2 &= v_2 h_2, & V &= V_1 + V_2.
\end{align*}
$$

(6.2)

Following the derivation of the vertically integrated continuity equation (5.11) we obtain

$$
\begin{align*}
\frac{\partial \xi}{\partial t} &= \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\left[ \frac{\partial}{\partial x} \left( u_1 h_1 + u_2 h_2 \right) + \frac{\partial}{\partial y} \left( v_1 h_1 + v_2 h_2 \right) \right], \\
\frac{\partial \eta}{\partial t} &= \frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} = -\left[ \frac{\partial}{\partial x} \left( u_2 h_2 \right) + \frac{\partial}{\partial y} \left( v_2 h_2 \right) \right].
\end{align*}
$$

(6.3), (6.4)

**b) pressure gradients and momentum equations**

Hydrostatic pressure is written in the upper layer

$$
p_1 = p_a + g \rho_1 \xi - g \rho_1 z
$$

(6.5)

and in the lower layer

$$
p_2 = p_a + g \rho_2 \xi - g (\rho_2 - \rho_1) (h_1 - \eta) - g \rho_2 z,
$$

(6.6)

where $p_a$ is atmospheric pressure on the surface and $\xi$ is surface elevation. Here the pressure in the lower layer has been calculated as $p_2 = p_a + g \rho_2 \int_{\eta}^{h_1} dz + g \rho_1 \int_{h_1 - \eta}^{\xi} dz$.

We get the pressure gradient forces as

$$
\begin{align*}
-\frac{1}{\rho_1} \frac{\partial p_1}{\partial x} &= -g \frac{\partial \xi}{\partial x}, & -\frac{1}{\rho_1} \frac{\partial p_1}{\partial y} &= -g \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial x} - g' \frac{\partial \eta}{\partial x}, \\
-\frac{1}{\rho_2} \frac{\partial p_2}{\partial x} &= -g \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial x} - g' \frac{\partial \eta}{\partial x}, & -\frac{1}{\rho_2} \frac{\partial p_2}{\partial y} &= -g \frac{\rho_2}{\partial y} - g' \frac{\partial \eta}{\partial y},
\end{align*}
$$

(6.7)

where

$$
g' = g \frac{\rho_2 - \rho_1}{\rho_2}
$$

(6.8)

is a reduced gravity.

Within the expressions (6.7), (6.8) we obtain the momentum equations for free (frictionless) motions analogically to (5.14), (5.15)
\[
\frac{\partial U_1}{\partial t} - fV_1 = -g h_1 \frac{\partial \xi}{\partial x}, \quad (6.9)
\]
\[
\frac{\partial V_1}{\partial t} + fU_1 = -g h_1 \frac{\partial \xi}{\partial y}, \quad (6.10)
\]
\[
\frac{\partial U_2}{\partial t} - fV_2 = -g h_2 \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial x} - g' h_2 \frac{\partial \eta}{\partial x}, \quad (6.11)
\]
\[
\frac{\partial V_2}{\partial t} + fU_2 = -g h_2 \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial y} - g' h_2 \frac{\partial \eta}{\partial y}, \quad (6.12)
\]

The momentum equations (6.9)-(6.12) together with the continuity equations (6.3)-(6.4) form a full set of equations for the motions of a two-layer fluid. They can be also written as

\[
\frac{\partial U}{\partial t} - fV = -gH \frac{\partial \xi}{\partial x} - g' h_2 \frac{\partial \eta}{\partial x}, \quad (6.13)
\]
\[
\frac{\partial V}{\partial t} + fU = -gH \frac{\partial \xi}{\partial y} - g' h_2 \frac{\partial \eta}{\partial y}, \quad (6.14)
\]
\[
\frac{\partial \xi}{\partial t} = -\left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right), \quad (6.15)
\]
\[
\frac{\partial U_2}{\partial t} - fV_2 = -g h_2 \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial x} - g' h_2 \frac{\partial \eta}{\partial x}, \quad (6.16)
\]
\[
\frac{\partial V_2}{\partial t} + fU_2 = -g h_2 \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial y} - g' h_2 \frac{\partial \eta}{\partial y}, \quad (6.17)
\]
\[
\frac{\partial \eta}{\partial t} = \left( \frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} \right), \quad (6.18)
\]

where \( H = h_1 + \frac{\rho_1}{\rho_2} h_2 \approx h_1 + h_2 \).

### 6.2. Splitting of long surface and internal waves

Consider one-dimensional motions without Coriolis force. The equations (6.13)-(6.18) are simplified as

\[
\frac{\partial U}{\partial t} = -gH \frac{\partial \xi}{\partial x} - g' h_2 \frac{\partial \eta}{\partial x}, \quad (6.19)
\]
\[ \frac{\partial \xi}{\partial t} = -\frac{\partial U}{\partial x} , \]
\[ (6.20) \]

\[ \frac{\partial U_2}{\partial t} = -gh_2 \frac{\rho_1}{\rho_2} \frac{\partial \xi}{\partial x} - g' h_2 \frac{\partial \eta}{\partial x} , \]
\[ (6.21) \]

\[ \frac{\partial \eta}{\partial t} = -\frac{\partial U_2}{\partial x} . \]
\[ (6.22) \]

In case of constant layer thickness, one 4-th order equation for the interface elevation can be derived

\[ \frac{\partial^4 \eta}{\partial t^4} - gH \frac{\partial^4 \eta}{\partial t^2 \partial x^2} + gg'h_2 \frac{\partial^4 \eta}{\partial x^4} = 0 . \]
\[ (6.23) \]

Searching for the wave solution

\[ \eta = \eta_0 e^{ikx - \omega t} , \]
\[ (6.24) \]

we obtain from (6.23)

\[ \omega^4 - gH \omega^2 k^2 + gg'h_2 k^4 = 0 . \]
\[ (6.25) \]

In case of small density difference

\[ \rho_2 - \rho_1 \ll \text{max}(\rho_1, \rho_2) \quad \Rightarrow \quad \frac{g'h_2}{H} \ll gH \]
\[ (6.26) \]

the quadratic equation for \( \omega^2 \) is simplified using Taylor expansion around \( \frac{gH}{2} \) and we obtain

\[ \omega^2 = \left[ \frac{gH}{2} \pm \left( \frac{gH}{2} - g' \frac{h_2}{H} \right) \right] k^2 . \]
\[ (6.27) \]

Due to (6.25), the barotropic (external) and baroclinic (internal) mode split within (6.27) into two different dispersion relations

\[ \omega_e = \sqrt{gH} k , \]
\[ (6.28) \]

\[ \omega_i = \sqrt{g' \frac{h_2}{H}} k = \sqrt{g' h_*} k , \]
\[ (6.29) \]
where \( h_e = \frac{h_1 h_2}{h_1 + h_2} \) is effective depth (effective layer thickness) and \( g' \) is reduced gravity.

Long internal waves are non-dispersive in a non-rotational fluid due to (6.29). Their dispersion relation is analogical to that of the long surface waves (6.28). Instead of \( c_e = \sqrt{gH} \) the phase speed of long internal waves is given as

\[
c_i = \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} \frac{h_1 h_2}{h_1 + h_2}} = \sqrt{g' h_e} \quad .
\]

Taking typical values for the Baltic Sea thermocline \( \rho_1 = 1.005 \text{ kg/m}^3, \rho_2 = 1.006 \text{ kg/m}^3, h_1 = 20 \text{ m}, h_2 = 60 \text{ m} \) we obtain \( c_e = 28 \text{ m/s}, c_i = 0.38 \text{ m/s} \).

The surface elevation wave pattern

\[
\xi = \xi_0 e^{i(kx - \omega t)}
\]

is coherent with the interface pattern (6.24), being in the same phase for the surface waves and in the opposite phase for the internal waves. The ratios of the amplitudes \( \xi_0 \) and \( \eta_0 \) are quite different for the external and internal mode. Indeed, from (6.21)-(6.22) we obtain

\[
\frac{\partial^2 \eta}{\partial t^2} - g h_2 \frac{\rho_1}{\rho_2} \frac{\partial^2 \xi}{\partial x^2} - g' h_2 \frac{\partial^2 \eta}{\partial x^2} = 0 \quad \Rightarrow \quad \omega^2 \eta_0 - k^2 \left( g h_2 \frac{\rho_1}{\rho_2} \xi_0 - g' h_2 \eta_0 \right) = 0
\]

and consequently from (6.28) we get for the surface waves

\[
B_e = \frac{\xi_0}{\eta_0} = \frac{\rho_2}{\rho_1} \frac{g(h_1 + h_2)}{g h_2} \approx \frac{h_1 + h_2}{h_2}
\]

and from (6.29) for the internal waves

\[
B_i = \frac{\xi_0}{\eta_0} = -\frac{\rho_2}{\rho_1} \frac{g'}{g} \frac{h_2}{h_1 + h_2} \approx -\frac{\Delta \rho}{\rho} \frac{h_2}{h_1 + h_2}
\]

In the internal mode maximum vertical velocities appear on the interface (Fig. 6.2) and decay linearly towards the bottom and the surface. The velocities/volume transports are shifted by \( \frac{\pi}{2} \) from the phase of the interface and the free surface.

Note that due to the mode splitting, the dispersion relation for the internal waves (6.29) can be obtained also in a “rigid lid” approach what means

\[
\frac{\partial \xi}{\partial t} = 0 \quad ; \quad U = 0 \quad ; \quad U_i = -U_2.
\]
It is important to notice that in case of (6.35) barotropic surface waves are filtered out but the water level spatial gradients are still non-zero since from (6.19) we obtain \( \frac{\partial \xi}{\partial x} = -\frac{g'h_2}{gH} \frac{\partial \eta}{\partial x} \).

This means that the water level changes slowly with frequency characteristics of baroclinic motions and fast barotropic motions are not excited. “Rigid lid” is strictly valid with constant layer thickness when barotropic-baroclinic coupling does not appear. However, many ocean circulation models use “rigid lid” in areas of complex topography and stratification with remarkable success.

**Figure 6.2.** A scheme of internal waves in a two-layer fluid.

### 6.3. Internal Poincare and Kelvin waves

We have shown that “rigid lid” does not distort one-dimensional long internal waves in case of constant layer thickness. Using “rigid lid” \( \frac{\partial \xi}{\partial t} = 0 \) to the initial two-dimensional equations (6.13)-(6.18), \( U = 0, V = 0 \) is valid only in the case of constant depth \( H = \text{const} \) and layer thickness \( h_2 = \text{const} \). Indeed, from the momentum equations (6.13), (6.14) we obtain with non-divergent transports \( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \) the vorticity equation

\[
\frac{\partial}{\partial t} \left( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) = g \left( \frac{\partial H}{\partial x} \frac{\partial \xi}{\partial y} - \frac{\partial H}{\partial y} \frac{\partial \xi}{\partial x} \right) - g \left( \frac{\partial h_2}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial h_2}{\partial y} \frac{\partial \eta}{\partial x} \right).
\]  

(6.36)

With \( H = \text{const} \) and \( h_2 = \text{const} \) the right side of (6.36) is exactly zero and vorticity \( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \) is not produced. Otherwise changing vorticity means non-zero transports.

From (6.13), (6.14) we obtain in case of \( U = 0, V = 0 \) the relations \( \frac{\partial \xi}{\partial x} = -\frac{g'h_2}{gH} \frac{\partial \eta}{\partial x} \), \( \frac{\partial \xi}{\partial y} = -\frac{g'h_2}{gH} \frac{\partial \eta}{\partial y} \) and the mathematical problem is reduced to
Following the derivation of the dispersion relation of two-dimensional rotational surface waves (5.46)-(5.53) we obtain in case of constant layer thickness analogically to (5.54).

\[ \omega^2 = f^2 + g'h_r(k^2 + l_n^2) \]  

(6.40)

The constant-depth water level equation (5.67) is in the “rigid lid” approach and constant layer thickness converted to the corresponding interface level equation

\[ \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \eta - g'h_r \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) = 0 \]  

(6.41)

Following the derivations in sub-chapters 5.6-5.8 we obtain dispersion relation for the internal (baroclinic) Poincare waves

\[ \omega_n = \pm \sqrt{f^2 + g'h_r(k^2 + l_n^2)} \]  

(6.42)

or in a non-dimensional form

\[ \left| \frac{\omega_n}{f} \right| = \sqrt{1 + \frac{g'h_r}{f^2}(k^2 + l_n^2)} = \sqrt{1 + R_d^2(k^2 + l_n^2)} \]  

(6.43)

where \( l_n \) is the cross-channel and \( k \) is the along-channel wave number. The phase speed can be expressed as

\[ c_n = \pm \sqrt{g'h_r + \frac{f^2}{k^2} \left( 1 + R_d^2 l_n^2 \right)} \]  

(6.44)

or in a non-dimensional form

\[ \left| \frac{c_n}{\sqrt{g'h_r}} \right| = \sqrt{1 + \frac{1}{k^2 R_d^2} + \frac{l_n^2}{k^2}} \].  

(6.45)

Here
Dynamical Oceanography

\[ R_d = \frac{\sqrt{g'h_*}}{f} \]  \hspace{1cm} (6.46)

is baroclinic (internal) Rossby deformation radius. Taking as above typical values for the Baltic Sea thermocline \( \rho_1 = 1.005 \text{ kg/m}^3 \), \( \rho_2 = 1.006 \text{ kg/m}^3 \), \( h_1 = 20 \text{ m} \), \( h_2 = 60 \text{ m} \) we obtain \( c_e = 28 \text{ m/s}, \ c_i = 0.38 \text{ m/s} \) and with \( f = 1.25 \cdot 10^{-4} \text{ s}^{-1} \) the value of baroclinic deformation radius is \( R_d = \frac{0.38}{1.25 \cdot 10^{-4}} = 3.04 \text{ km} \) while the barotropic deformation radius is \( \frac{28}{1.25 \cdot 10^{-4}} = 224 \text{ km} \).

Baroclinic Kelvin waves in channel of constant layer thickness also follow the corresponding barotropic expressions. With \( gH \rightarrow g'h_* \) we obtain from (5.90) dispersion relation

\[ \omega = \pm \sqrt{g'h_*}k \]  \hspace{1cm} (6.47)

Consequently the baroclinic Kelvin waves have a non-dispersive phase speed \( c = \pm \sqrt{g'h_*} \) independent from the wave number.

The amplitude of baroclinic Kelvin wave decays from the channel wall seawards as \( \exp\left(-\frac{y}{R_d}\right) \) and the wave pattern traveling near the wall \( y = 0 \) (sea is located at \( y > 0 \)) along the \( x \)-axis is

\[ \eta = \eta_0 \exp\left(-\frac{y}{R_d}\right) \cos\left[k(x - \sqrt{g'h_*}t)\right]. \]  \hspace{1cm} (6.48)

Since the value of baroclinic \( R_d \) is from a few to a few tens of kilometers, baroclinic Kelvin waves are quite strongly concentrated near the coast. Similar to the barotropic Kelvin waves, the cross-channel currents in the both layers are exactly zero. Along-channel currents in the upper and lower layer are in the opposite phase. In practice, coastal jet currents are frequently observed in a band of the width \( R_d \). When the coastal slope is wider than \( R_d \) then coastal-trapped waves occur.

6.4. Baroclinic topographic Rossby waves in a thin near-bottom layer

Note that baroclinic topographic waves in a two-layer ocean cannot be handled in the approach of reduced interface elevation equation (6.41). A two-layer analogue of the variable-depth water level equation (5.63) can be derived if one layer is much thicker than another layer.
Following the sub-chapter 5.10, consider a channel with sloping bottom, then \( h_1 = \text{const} \) and 
\[
h_2 = \bar{h}_2 \left( 1 - \frac{\alpha y}{L} \right)
\]
with small bottom slope \( \alpha << 1 \), \( \frac{\partial h_2}{\partial y} = -\bar{h}_2 \frac{\alpha}{L} << 1 \). In case of thin near-bottom layer \( h_2 << h_1 \) the effective depth is
\[
h_* = \frac{h_1 h_2}{h_1 + h_2} \approx \bar{h}_2 \left( 1 - \frac{\alpha y}{L} \right),
\]
where effective slope is \( \alpha \approx \alpha \frac{h_1}{h_1 + h_2} \). Analogically to (5.108) the interface level equation takes a form
\[
\frac{\partial}{\partial t} \left[ \left( \frac{\partial^2}{\partial t^2} + f^2 \right) \eta - g \bar{h}_2 \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) - \frac{\alpha}{L} \frac{\partial \bar{h}_2}{\partial y} \frac{\partial \eta}{\partial y} - \frac{\alpha}{L} \frac{\partial \bar{h}_2}{\partial x} \frac{\partial \eta}{\partial x} \right] = 0.
\]
In the solution procedure, again the Poincare and Kelvin modes of the equation (6.50) get only very little modified by the bottom slope since the cross-channel additional decay term proportional to \( \exp \left( \frac{\alpha y}{L} \right) \) is very small. In the low frequency band \( \omega^2 << \frac{\alpha f g \bar{h}_2}{L \omega} \) we obtain analogically to (5.120) the dispersion relation for the baroclinic topographic waves
\[
\omega = -\frac{\alpha f}{L} \frac{k}{k^2 + l_n^2 + \frac{1}{R_d^2}} \quad n = 1, 2, \ldots
\]
where \( l_n \) is the cross-channel and \( k \) is the along-channel wave number, \( L \) is the channel width and \( R_d = \frac{\sqrt{g \bar{h}_2}}{f} \) is the baroclinic deformation radius. We note again that at large wave numbers (short wavelengths) \( k \gg \frac{1}{R_d} \) the gravity acceleration apparent in \( R_d \) has no influence and the dispersion type \( \omega \sim \frac{1}{k} \) is controlled by conservation of potential vorticity.

With small wave numbers \( k \ll \frac{1}{R_d} \) the dispersion type is \( \omega \sim k \). While the term “internal waves” is reserved for gravitational waves, topographic waves as vorticity waves do not belong to that category.

Further in Chapter 8, considering the topographic waves in a continuously stratified fluid, the absolute bottom slope \( \frac{\partial h}{\partial y} = -\bar{\alpha} \) is used. In order to allow better comparison, the dispersion relations for the barotropic (5.120) and the two-layer baroclinic (6.51) topographic waves can be written as
\[ \omega = -\frac{\tilde{\alpha} f}{h} \frac{k}{k^2 + l_n^2 + \frac{1}{R_d^2}} \quad n = 1, 2, \ldots \]  

(6.52)

where the cases are

1) barotropic: \( h \) - bottom depth, \( R_d \) - barotropic Rossby deformation radius;

2) baroclinic: \( h \) - near-bottom layer thickness, \( R_d \) - baroclinic Rossby deformation radius.

We see that in the stratified fluid both \( h \) and \( R_d \) are smaller than in the non-stratified sea, resulting in higher frequencies and longer periods in the large wave number (short wave length) band. While the stratification is present, splitting of barotropic and baroclinic modes does not take place.

It is interesting to notice that dispersion relation of topographic Rossby waves (6.52) can be easily obtained from the quasi-geostrophic equation (will be in more detail considered later) as a simplification from (6.50)

\[ \frac{\partial}{\partial t} \left( \Delta \eta - \frac{\eta}{R_d^2} \right) + \frac{\tilde{\alpha} f}{h} \frac{\partial \eta}{\partial x} = 0. \]  

(6.53)