NOTES ON SMALL-SCALE PROCESSES AND MIXING IN THE OCEAN

These lectures may be considered as an introduction to the subject for graduate students training in physical oceanography. The following topics are concerned: dimensional analysis and self-similarity, small-scale turbulence and turbulent diffusion, microstructure, double-diffusive convection, thermohaline finestructure, vertical and horizontal mixing.

LECTURE 1. Introduction to dimensional analysis and self-similarity

Dimensional analysis is a powerful tool to obtain quantitative description of processes and phenomena in physics and geophysical dynamics in particular (especially in turbulence studies).

The essence of dimensional analysis can be expressed in the form of II-theorem:

Suppose that there is a physical dependence of a dimensional variable $a_0$ on $n$ dimensional governing parameters $a_1, a_2, ..., a_n$

$$a_0 = f(a_1, a_2, ..., a_n)$$

(*)

consisting of $k \leq n$ independent dimensions. If $k < n$, the dependence (*) can be expressed as a dependence of non-dimensional power combination of $a_0$ and the governing parameters on $(n - k)$ independent, non-dimensional power combination of the governing parameters

$$\frac{a_0}{a_1^{a_1}...a_n^{a_n}} = \Phi\left(\frac{a_k^{a_k}}{a_1^{a_1}...a_n^{a_n}}, \frac{a_l^{a_l}}{a_1^{a_1}...a_n^{a_n}}, ..., \frac{a_n^{a_n}}{a_1^{a_1}...a_n^{a_n}}\right)$$

If $k = n$, then $\Phi = C = \text{const} \neq 0$, and $a_0$ is uniquely expressed as a power combination of the governing parameters

$$a_0 = Ca_1^{a_1}...a_n^{a_n}.$$
If we hang a heavy body of mass $m$ on a weightless thread of a length $l$ and push the body slightly, it will oscillate around its equilibrium position (Fig. 1.1).

If the friction is neglected and oscillation amplitude $x_{\text{max}}$ is small ($\alpha_{\text{max}} \approx x_{\text{max}}/l \ll 1$), the period of pendulum oscillations $T$ can be expressed as

$$T = 2\pi \sqrt{\frac{l}{g}}$$

(1.1)

where $g$ is the gravity acceleration.

Formula (1.1) is known from high school course of physics where it was derived using dynamical consideration of pendulum. However, we can arrive to similar formula based on dimensional analysis.

Since the thread is weightless, oscillation amplitude is small, and friction is neglected, the period $T$ can depend on $m$, $l$, and $g$. Then, since both forces describing the balance of pendulum (gravity force $mg$ and inertia force $m \cdot dx/dt$) are proportional to $m$, we can assert that $T$ does depend on $l$ and $g$ only.

Because in our case $n = k = 2$, using $\Pi$-theorem we can finally write a power-like form

$$T = C \cdot l^\beta \cdot g^\gamma$$

(1.2)

where $C$ is some constant of the order of unity, and determine power indices $\beta$ and $\gamma$ from the condition of dimension identity for left and right sides of (1.2):

$$[T] = s, [l] = m, [g] = m \cdot s^2$$

$$0 = \beta + \gamma \Rightarrow \gamma = -\frac{1}{2}, \beta = \frac{1}{2}$$

Therefore

$$T = C \cdot l^{0.5} g^{-0.5} = C \cdot \sqrt{\frac{l}{g}}$$

(1.2’)

Of course, the value of constant $C$ cannot be found from dimensional analysis.

More generally the dimensional consideration for the pendulum case can be formulated as follows. Pendulum period $T$ in frictionless case may depend on $l$, $g$, $m$, oscillation amplitude $x_{\text{max}}$, and mass of the thread $M_s$, so that $n = 5$, $k = 3$ the dependence to be found can be expressed in the following, most generalized dimensionless form

$$\frac{T}{l^{0.5} g^{-0.5}} = f\left(\frac{x_{\text{max}}}{l}, \frac{M_s}{m}\right)$$

(1.3)

where $f$ is some dimensionless function of two dimensionless parameters, $x_{\text{max}}/l$ and $M_s/m$.

If we suggest that there is some non-zero asymptotic value of $f$ at $x_{\text{max}}/l \rightarrow 0$ and $M_s/m \rightarrow 0$ (i.e., $\lim(f) = C \neq 0$ at $x_{\text{max}}/l \rightarrow 0$ and $M_s/m \rightarrow 0$, we arrive at (1.2’) once again.

1.2. Pythagore formula

It is a fun that the Pythagore formula being well known from middle school geometry course can be re-derived from dimensional analysis (see Fig. 1.2).

It is obvious that the area of a rectangular triangle can be uniquely defined by the value of hypotenuse $c$ and smaller angle $\phi$, so
using dimensional reason we can write for the “big” triangle in Fig. 1.2:

\[ S = c^2 \cdot f(\phi) \]  

(1.4)

where \( f \) is some function of angle \( \phi \).

Expressions similar to (1.4), of course, will be valid for areas \( S_1 \) and \( S_2 \) of two smaller rectangular triangles

\[ S_1 = a^2 \cdot f(\phi), \quad S_2 = b^2 \cdot f(\phi) \]  

(1.5)

It is obvious that

\[ S = S_1 + S_2 \]  

(1.6)

If we now substitute expressions (1.4) and (1.5) into (1.6), we obtain

\[ c^2 \cdot f(\phi) = a^2 \cdot f(\phi) + b^2 \cdot f(\phi) \implies c^2 = a^2 + b^2 \]

so that the Pythagorean formula is proved from dimensional analysis.
1.3. Atomic bomb explosion

After the first test of atomic bomb in 1945 a film showing propagation of fire ball caused by the explosion was presented to public, while the energy released by the explosion was considered as a top secret information. However, soon after the film show, Sir Jeffrey Taylor, famous English scientist, published a paper where the energy of atomic bomb explosion was correctly estimated (which was regrettable to US officials).

Sir Jeffrey disclosed the top secret by means of dimensional analysis. He suggested that at the first stage (just after the explosion) the growth of the fire ball radius (i.e., propagation of the shock wave front) $r_f$ is determined by the energy released, $E$, the undisturbed air density $\rho_0$, and time $t$, (undisturbed air pressure is of no importance, since it is a factor 100000 smaller than the pressure inside the fire ball):

$$r_f = r_f(E, t, \rho_0)$$

(1.7)

Since $[r_f] = m$, $[E] = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$, $[t] = \text{s}$, $[\rho_0] = \text{kg} \cdot \text{m}^{-3}$, we have $n = k = 3$, and $\Pi$-theorem implies a unique power dependence

$$r_f = C \cdot (E \cdot t^2 / \rho_0)^{1/5}$$

(1.8)

where $C$ is a constant of the order of unity.

Relationship (1.8) can be re-written as

$$(5/2) \cdot \lg r_f = (1/2) \cdot \lg (C^5 \cdot E / \rho_0) + \lg t$$

(1.8')

Fig. 1.3 shows empirical dependence of $(5/2) \cdot \lg r_f$ on $\lg t$ obtained by means of processing of the film.

Fig.1.3. Empirical dependence of $(5/2) \cdot \lg r_f$ on $\lg t$ obtained by means of processing of the film.

Using data presented in Fig. 1.3 and taking $C = 1$, Taylor estimated the energy of atomic bomb explosion at $E = 10^{14} \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$.

**Conclusion:** don’t spy but think!
1.4. Propagation of gravity waves at a density interface
(c.e.g. at the sea surface)

\[ L - \text{wave length} \]
\[ k = \frac{2\pi}{L} - \text{wave number} \]
\[ T - \text{wave period} \]
\[ \omega = \frac{2\pi}{T} - \text{wave frequency} \]
\[ h(t,x) - \text{wave elevation} \]
\[ H - \text{sea depth/ lower layer thickness} \]
\[ \rho_w - \text{water/ lower layer density} \]
\[ \rho_a - \text{air/ upper layer density} \]
\[ g - \text{gravity acceleration} \]
\[ h_{\text{max}} - \text{wave amplitude} \]

Wave function
\[ h(x,t) = f(kx - \omega t) \]
where \( f(\phi) = f(2\pi + \phi) \) is a periodic function with period of \( 2\pi \)
[e.g., \( f = A\sin(\phi + \phi_0) \)]

Let us find expression for phase velocity of the wave \( c_f \equiv \frac{\omega}{k} \).

The phase velocity is the rate of propagation of wave crests and troughs.
\[ c_f = c_f(k,H,\rho_w,\rho_a,g,h_{\text{max}}) \] (1.9)

We suggest that \( h_{\text{max}}k << 1 \Rightarrow \) for small (linear) waves the phase velocity does not depend
on the wave amplitude (\( h_{\text{max}} \) can be excluded from governing parameters).

In gravity waves the inertia force is balanced by the gravity (buoyancy force). Since the
inertia force is proportional to \( \rho_w \) (the mass of unit volume) and the buoyancy force is
proportional to \( g(\rho_w - \rho_a) \) we can use a combination \( g(\rho_w - \rho_a)/\rho_w \) as a governing
parameter instead three parameters: \( g, \rho_w, \rho_a \).

Since \( \rho_w >> \rho_a \), we can write: \( \frac{\rho_w - \rho_a}{\rho_w} \approx g \). Therefore, only three parameters are
important:
\[ c_f = c_f(k,g,H) \] (1.9')

Two asymptotics can be considered: short and long waves

1.4.a. Short waves
If \( kH >> 1 \) (or \( H >> L \)) the wave motion is located in the near-surface layer and vanishes
in the deeper layer. Therefore, the sea depth, \( H \), does not affect the wave motion, and \( H \) is no
longer a governing parameter, and (1.9') is reduced to
\[ c_f = c_f(k,g) \] (1.9'')

Since \( [c_f] = m \cdot s^{-1}, [k] = m^{-1}, [g] = m \cdot s^{-2} \), dimensional considerations yield
\[ c_f = C(g/k)^{1/2} \] (1.10)

“Dynamical” considerations show that \( C = 1 \) so (1.10) is re-written as \( c_f = (g/k)^{1/2} \) or
\( \omega = (gk)^{1/2} \).

1.4.b. Long waves
If $kH \ll 1$ (or $H \ll L$), the sea depth, $H$, becomes important for wave motion and the phase velocity is no longer depend on the wave number:

$$c_f = c_f(H, g)$$  \hspace{1cm} (1.9'')

Dimensional considerations result in

$$c_f = C(gH)^{1/2}$$  \hspace{1cm} (1.11)

and “dynamical” considerations give $C = 1$ so (1.11) is re-written as $c_f = (gH)^{1/2}$. Note that here the phase velocity does not depend on wave number/length: the long waves are free of dispersion.

1.5. Dispersion of a passive tracer from a localized instant source

Let us consider a problem of diffusion of a passive tracer from localized instant source. 1D case is considered for simplicity. Mathematically the problem can be formulated as

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial x} \left( \kappa \frac{\partial q}{\partial x} \right)$$  \hspace{1cm} (1.12)

where $q(x,t)$ is 1D concentration of tracer ($[q] = \text{kg/m}$ since the problem is one-dimensional), $x$ is spatial co-ordinate, $t$ is time, $\kappa = \text{const} > 0$ is kinematic diffusivity of the tracer ($[\kappa] = \text{m}^2/\text{s}$), $Q$ is the source capacity ($[Q] = \text{kg}$), and $\delta(x)$ is the delta-function defined as

$$\delta(x) = 0 \text{ at } x \neq 0, \quad \int_{-\infty}^{+\infty} \delta(x)dx = 1$$  \hspace{1cm} (1.14)

Concentration field to be calculated, $q(x,t)$, is a function of four governing parameters, $x, t, \kappa, Q$, which have three independent dimensions (kg, m, and s). Using $\Pi$-theorem ($n=4, k=3$) and introducing a time-dependent length scale $\tilde{x} = (\kappa t)^{1/2}$, the concentration field $q(x,t)$ can be expressed as a function of one independent non-dimensional parameter $\zeta$

$$q(x,t) = \frac{Q}{\tilde{x}} f\left(\frac{x}{\tilde{x}}\right) = \frac{Q}{(\kappa t)^{1/2}} f\left[ \frac{x}{(\kappa t)^{1/2}} \right] = \frac{Q}{(\kappa t)^{1/2}} f(\zeta), \quad \zeta = \frac{x}{(\kappa t)^{1/2}}$$  \hspace{1cm} (1.15)

where $C$ is a constant to be calculated from (1.14).

Calculating $t$ and $x$ derivatives of (1.15) and substituting them into (1.12) we obtain an ordinary differential equation for function $f(\zeta)$

$$2f'' + \zeta f' + f = 0$$  \hspace{1cm} (1.16)

where $f'$ and $f''$ are the first and second $\zeta$-derivatives of $f$.

Note that by introducing special units for length $(\kappa t)^{1/2}$ and concentration $Q/(\kappa t)^{1/2}$, the two-variable function $q(x,t)$ was reduced to a one-variable function $f(\zeta)$. In other words, we found a self-similarity of distributions $q(x,t)$. That is, if we measure concentration in units of $Q/(\kappa t)^{1/2}$ and length in units of $(\kappa t)^{1/2}$, the shape of spatial distribution of concentration at different value of time will be identical.

Solution of (1.16) which fits the problem is

$$f(\zeta) = \exp(-\zeta^2/4)$$  \hspace{1cm} (1.17)

Calculating constant $C$ from (1.14) we obtain finally

$$q(x,t) = \frac{Q}{2\pi^{1/2} (\kappa t)^{1/2}} \exp\left[-\frac{x^2}{4(\kappa t)^{1/2}}\right]$$  \hspace{1cm} (1.18)