LEcTURe 2. INTRODUCTION TO TURBuLENCe

Everyday life gives a lot of examples of turbulent flows: the smoke of a cigarette, motion of clouds, abrupt changes in direction and velocity of the wind, shaking a plane when you are advised to fasten seat-belts. Despite everybody “knows” intuitively what is the turbulence, it is not easy to define the subject. Probably the most simple definition is as follows:

A Simple Definition of Turbulence

A turbulent flow is a flow which is disordered in time and space and which is able to mix transported quantities much more rapidly than molecular diffusion alone.

Momentum and heat transfer in a turbulent flow differs much from that in a laminar flow, and it makes the study of turbulence extremely important in engineering, meteorology, oceanography, etc.

Navier-Stokes Equations

A laminar flow of incompressible, viscous fluid can be described by the Navier-Stokes and continuity equations

\[
\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + X_i + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1.1}
\]

\[
\frac{\partial u_i}{\partial x_i} = 0 \tag{1.2}
\]

where \( u_i \) is \( i \)-th velocity component, \( i = 1,2,3, (x_i, t) \) are Cartesian coordinates and time, \( p \) is pressure, \( \rho \) is density, \( \nu \) is kinematic viscosity, \( X_i \) is an external, mass force applied to the fluid (i.e., the gravity force). The first term in the left side of (1.1) describes nonstationarity of the flow (the time change), and the second one is a nonlinear inertia term. In the right side of (1.1) there are a pressure gradient force, some external force \( X_i \), and a viscosity force.
Adding to equations (1.1) and (1.2) an equation of temperature transport

\[ \frac{\partial T}{\partial t} + \frac{\partial u_j T}{\partial x_j} = \chi \frac{\partial^2 T}{\partial x_j \partial x_j} \]  

(1.3)

where \( T \) is temperature, \( \chi \) is thermal diffusivity coefficient, we get a closed system of five equations with five unknown variables \( u_1, u_2, u_3, p, T \) (\( \rho \) is taken constant for simplicity, so the temperature is considered as a passive tracer). This system can be solved provided that we prescribe proper boundary and initial conditions. Hence, we may say that the description of laminar flows is a matter of mathematics rather than physics. This is not the case if one is considering a turbulent flow.

**Transition to Turbulence, Reynolds Number**

Let us consider a flow with a typical velocity scale \( U \) and spatial scale \( L \). Using this scales, the nonlinear inertia term and the viscosity term in (1.1) can be estimated as

\[ \frac{\partial u_j u_i}{\partial x_j} \approx \frac{U^2}{L} \quad \text{and} \quad \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \approx \frac{U}{L^2} \]

Hence, the ratio of nonlinear inertia to viscosity terms can is estimated as \( UL/\nu \), and this nondimensional value is called the Reynolds number, \( Re \):

\[ Re = \frac{UL}{\nu} \]  

(1.4)

If \( Re >> 1 \), the viscosity term (1.1) may be neglected, and it looks as though a flow may be considered as inviscid one. But the problem is that just at \( Re >> 1 \) the flow becomes turbulent, e.g., consisting of a lot of eddies of different size which are random in space and time. The transition from laminar to turbulent flow takes place at \( Re = 600-2000 \). Hence, the flow becomes turbulent when nonlinear inertia term in the Navier-Stokes equations gets very much greater than the viscosity term.

**An Illustrating Example of Transition from Laminar to Turbulent Flow**

Let us consider a smoke of cigarette (see Fig. 1). Just above the cigarette, the smoke is ascending right upward, in a regular manner, so the flow is laminar. Then, at some distance \( L \) above the cigarette, the smoke path suddenly becomes irregular, i.e., turbulent. We can
visually estimate a velocity of the smoke ascend as \( U \approx 30 \text{ cm s}^{-1} \), and a distance at which the transition to turbulence takes place as \( L \approx 3 \text{ cm} \). Then, taking a value \( v_a = 0.14 \text{ cm}^2\text{s}^{-1} \) as the kinematic air viscosity we get an estimate of the Reynolds number

\[
\text{Re}_\text{cigarette smoke gets turbulent} = \frac{UL}{v_a} \approx \frac{30 \text{ cm s}^{-1} \times 3 \text{ cm}}{0.14 \text{ cm}^2\text{s}^{-1}} = 642
\]

The obtained value has just fallen into the above range of “critical” Reynolds numbers in which the laminar flow does become turbulent. Everyone can easily carry out this experiment to be convinced that the Reynolds criteria works well.

**Multiscale nature of turbulence. Vortex stretching: a way to isotropy**

Once the flow gets turbulent it can no longer be characterized by a single velocity and length scales. That is, different size eddies will be present having different Reynolds number estimates

\[
\text{Re}_{\text{eddy}} = \frac{U_{\text{eddy}}L_{\text{eddy}}}{v}
\]

where \( U_{\text{eddy}} \) and \( L_{\text{eddy}} \) are the velocity scale of an eddy and its size. Large eddies are expected to be characterized by a large value \( \text{Ri}_{\text{eddy}} \approx \text{Ri} \gg 1 \), while there will be eddies small enough to be characterized by a small value of \( \text{Ri}_{\text{eddy}} \approx 1 \). In these small eddies, viscous force cannot be neglected. The small eddies are of no importance if one is interested in effect of turbulence on the mean flow. Hence the effect of turbulence on the structure of the mean flow as well as a scalar property transfer (i.e. heat transfer) can be examined from equations (1.1) and (1.3) with no molecular viscosity and diffusivity terms.

Turbulent eddies have both translational and rotational motion. The net rates of rotation (or angular velocities) about the \( x, y, \) and \( z \) axes are

\[
\omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \quad \omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial w}{\partial x} \right), \quad \omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]

where \( u, v, w \) are the velocity components along \( x, y, z \) axes, respectively. A vorticity is determined as twice vector of the angular velocity: \((2\omega_x, 2\omega_y, 2\omega_z)\).

Let us consider a rotation an fluid element about \( z \) axis: \( \omega_z \neq 0 \). If in addition to this rotation the fluid element is under the influence of a rate of linear strain in the \( z \) direction, \( \frac{\partial w}{\partial z} > 0 \), the element will be stretched in \( z \) direction and its cross-section in the \( xy \)-plane will get smaller. Because the product of vorticity and the square of cross-section of the element will remain constant (provided that the viscous force is neglected; it follows from the conservation of angular momentum), during the stretching process the kinetic energy of rotation increases and the scale of the motion in the \( xy \)-plane decreases. Therefore the stretching in one direction (\( z \) direction here) can decrease the length scale and increase the velocity components in the other two directions (\( x \) and \( y \)) which in turn stretch other elements of fluid with vorticity components in these directions, and so on. The length scale of motions is getting smaller at each generation. If we draw out a “family tree” (Fig.1) showing how stretching in \( z \) direction
(the first generation) intensifies the motion in $x$ and $y$ directions, producing smaller-scale stretching in $x$ and $y$ directions (the second generation) and then intensifying motions and producing stretching in the $y, z$ and $x, x$ directions respectively (the third generation), and so on, we can see qualitatively that that an initial stretching in one direction produces nearly equal amounts of smaller-scale stretching in each of the $x, y, z$ directions after a few “generations” of the process.

Thus small-scale eddies in turbulence do forget the preferred orientation of the large-scale motion. The energy is transferred from larger to smaller eddies, and finally the energy of the smallest eddies (with $Ri_{edd} \approx 1$) is dissipated by viscosity into thermal internal energy or “heat”. The above scenario of developed turbulence is expressed neatly the following rhyme [Richardson, 1922])

Big whorls have little whorls,  
Which feed on their velocity;  
And little whorls have lesser whorls,  
And so on to viscosity

**Why stretching of eddies does increase the energy of turbulence**

Let us consider two initially identical eddies (“rolls”) rotating with an angular velocity $\omega(0)$ about the $x$-axis, the first one is subjected to stretching up ($\frac{\partial u}{\partial x} = s > 0$), and the second one is subjected to stretching down with the strain rate of the same absolute value ($\frac{\partial u}{\partial x} = -s < 0$), so these two events have the same probability (see Fig.3).
Fig. 3. Rolls stretching up and stretching down

The kinetic energy of a roll can be calculated as

\[ E = \rho L \int_0^R \frac{\omega^2 r^2}{2} 2\pi r dr = \pi \rho L \omega^2 \frac{R^4}{4} = \rho L \frac{1}{4\pi} (\omega S)^2 \]  

where \( S = \pi R^2 \) is the cross-section area of the roll. The conservation of angular momentum (provided the viscosity is neglected) requires that \( \omega S = \omega(0)S(0) = \text{const} \). Therefore, the energy change during stretching process is expressed as

\[ E(t) = E(0) \frac{L(t)}{L(0)} \]  

Since

\[ \frac{dL}{dt} = \frac{\partial u}{\partial x} L \]  

we get

\[ L(t) = L(0) \exp \left( \frac{\partial u}{\partial x} t \right) \]  

and the sum of energies of a pair of stretching up roll and stretching down roll is

\[ E_{\text{sum}}(t) = E_{\text{sum}}(0) \cosh(st) = E_{\text{sum}}(0) \frac{\exp(st) + \exp(-st)}{2} \]  

Therefore, we may conclude that the process of stretching of eddies does increase kinetic energy of turbulence.

An Advanced Definition of Turbulence

Turbulence is a three-dimensional time-dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between a minimum determined by viscous forces and a maximum determined by boundary condition of the flow. It is the usual state of fluid motion except at low Reynolds numbers.

Reynolds Equations

To describe statistically the effect of turbulence on the mean flow, let us decompose velocity and other quantities into mean values and turbulent fluctuation

\[ u_i = \bar{u}_i + u'_i, \ p = \bar{p} + p', \ T = \bar{T} + T' \]  

where overbar denotes the mean value, and (’) denotes the fluctuation. The averaging rules are as follows:

\[ \bar{\bar{a}} = \bar{a}, \ \bar{a}' = 0 \]
were \( a \) is any quantity to be averaged. Substituting (1.6) into (1.1)-(1.4) and applying the averaging to all terms of these equations we get

\[
\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} + \frac{\partial \overline{u_i'}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \overline{\rho}}{\partial x_i} + \overline{X_j} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}
\]

(1.1′)

\[
\frac{\partial \overline{u_i}}{\partial x_i} = 0
\]

(1.2′)

\[
\frac{\partial \overline{T}}{\partial t} + \frac{\partial \overline{u_i T'}}{\partial x_j} + \frac{\partial \overline{u_i' T'}}{\partial x_j} = \chi \frac{\partial^2 \overline{T}}{\partial x_j \partial x_j}
\]

(1.3′)

Equations (1.1′)-(1.3′) are called the Reynolds equations, and tensor \( \overline{u_i' u_j'} \) is called the Reynolds stress tensor. Unlike equations (1.1)-(1.4), the system of equations (1.1′)-(1.4′) is no longer closed because a number of new variables, namely correlations \( \overline{u_i' u_j'} \) and \( \overline{u_i T'} \), \( i, j = 1, 2, 3 \) have been added while the number of equations did not changed. To close this system, using the above equations one can derive some new equations for \( \overline{u_i' u_j'} \) and \( \overline{u_i T'} \)

\[
\frac{\partial \overline{u_i' u_j'}}{\partial t} = \dots (1.7)
\]

\[
\frac{\partial \overline{u_i' T'}}{\partial t} = \dots (1.8)
\]

(To derive equations (1.7) one can formulate \( \frac{\partial \overline{u_i' u_j'}}{\partial t} = u_i' \frac{\partial \overline{u_j'}}{\partial t} + u_j' \frac{\partial \overline{u_i'}}{\partial t} \), substitute expression for \( \frac{\partial \overline{u_i'}}{\partial t} \) taken from (1.1), (1.1′), and (1.6) and then make the averaging. The same is valid for (1.8). For details, see Monin and Yaglom, 1971, p.374.)

Unfortunately, these new equations, (1.7)-(1.8), will contain in their right sides some more new terms, namely triple correlation terms like \( \overline{p' u_i' u_j'} \), \( \overline{u_i' u_j' u_k'} \) or so, and the system will remain unclosed. Therefore, statistical studies of the equations of motion always lead to a situation in which there are more unknowns than equations. This is called the closure problem of the turbulence theory: one has to make (very often ad hoc) assumptions to make the number of equations equal to number of unknowns. If one considers equations (1.1′)-(1.3′) and do not consider equations (1.7)-(1.8) and makes assumptions about \( \overline{u_i' u_j'} \) and \( \overline{u_i' T'} \) this is called the first moment closure scheme. If one considers equations (1.1′)-(1.3′), (1.7)-(1.8) and makes assumptions about triple correlation this is called the second moment closure
One of the most powerful tools to close equations of turbulent motion is dimensional analysis.

**Eddy viscosity and diffusivity.**

Let us consider for simplicity a flat, horizontally uniform mean flow directed along the $x$ axis: $\bar{u} = \bar{u}(z, t)$, $\bar{T} = \bar{T}(z, t)$ $\bar{v} = 0$, $\bar{w} = 0$, $\partial \bar{u} / \partial x = 0$, $\partial \bar{u} / \partial y = 0$. In this case equations (1.1') and (1.3') reduce to

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial \bar{u}'w'}{\partial z} + \nu \frac{\partial^2 \bar{u}}{\partial z^2}$$

$$\frac{\partial \bar{T}}{\partial t} = -\frac{\partial \bar{w}'T'}{\partial z} + \chi \frac{\partial^2 \bar{T}}{\partial z^2}$$

The last term in the right side of (1.9) and (1.10) is the convergence ($-\partial / \partial z$) of vertical fluxes of momentum ($-\nu \partial \bar{u} / \partial z$) and temperature ($-\chi \partial \bar{T} / \partial z$) due to molecular viscosity and heat diffusivity. Similarly, $\bar{u}'w'$ and $\bar{w}'T'$ are vertical fluxes of momentum and temperature due to turbulent eddies. Using the analogy with molecular viscosity and diffusivity, we can write

$$\bar{u}'w' = -K_M \frac{\partial \bar{u}}{\partial z}, \quad \bar{w}'T' = -K_H \frac{\partial \bar{T}}{\partial z}$$

where $K_M$ and $K_H$ are new variables called the coefficient of eddy viscosity (of turbulent viscosity) and coefficient of eddy diffusivity for heat, respectively. Boussinesq (1877) was the first to suggest (1.11). Using (1.11) we can rewrite (1.9) and (1.10) as

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial z} \left( K_M \frac{\partial \bar{u}}{\partial z} \right) + \nu \frac{\partial^2 \bar{u}}{\partial z^2}$$

$$\frac{\partial \bar{T}}{\partial t} = \frac{\partial}{\partial z} \left( K_H \frac{\partial \bar{T}}{\partial z} \right) + \chi \frac{\partial^2 \bar{T}}{\partial z^2}$$

However, to close the equations of motion, we have to suggest some expressions or equations for $K_M$ and $K_H$.

**An example of first moment closure. Turbulent flow near the rigid wall**

In accordance with (1.9) and (1.9'), a stationary turbulent flow near the wall at $z = 0$ is described by equation

$$-\frac{\partial \bar{u}'w'}{\partial z} + \nu \frac{\partial^2 \bar{u}}{\partial z^2} = \frac{\partial}{\partial z} \left( K_M \frac{\partial \bar{u}}{\partial z} \right) + \nu \frac{\partial^2 \bar{u}}{\partial z^2} = 0$$

Integrating (1.12) over $z$, we get

$$\tau(z) = -\rho \bar{u}'w' + \rho \nu \frac{\partial \bar{u}}{\partial z} = \tau_0 = \text{const}$$
where $\tau(z)$ is the total (turbulent+viscous) shear stress and $\tau_0$ is the viscous shear stress on the wall (the turbulent stress, $-\rho u'w'$, on the wall vanishes because $w' = 0$ here). Actually we cannot determine the profile of velocity from (1.13). Nevertheless, the possible form of the function $\bar{u}(z)$ may be obtained by dimensional analysis. We may suppose that the ability of a wall to resist or decelerate the flow depends on the height of protrusions on it, $h_0$ (see Fig.4).

In accordance with (1.13), we may expect that the mean velocity profile $\bar{u}(z)$ will depend on $z$, $h_0$, $\nu$, and a friction velocity $u_*$ defined as

$$u_* = \left( \frac{\tau_0}{\rho} \right)^{1/2}$$  \hspace{1cm} (1.14)

That is we have three length scales in the problem:

$$z, z_* = \frac{\nu}{u_*}, \text{ and } h_0$$  \hspace{1cm} (1.15)

where $z_*$ is called the friction length. Hence, from dimensional analysis the dependence of $\bar{u}(z)$ on $z$, $\tau_0$, $h_0$, $\rho$, and $\nu$ may be written in a general form

$$\bar{u}(z) = u_* f \left( \frac{zu_*}{\nu}, \frac{h_0 u_*}{\nu} \right)$$  \hspace{1cm} (1.16)

If $h_0 \ll z_*$ the protrusions will not affect the mean flow at $z \gg h_0$, and (1.16) reduces to

$$\bar{u}(z) = u_* f \left( \frac{zu_*}{\nu}, z, z_* \gg h_0$$  \hspace{1cm} (1.16')

The important result (1.16') in turbulence theory is called the universal law of the wall; it was first formulated by Prandtl (1925). The case $h_0 \ll z_*$ is referred as dynamically smooth wall, and the case $h_0 \gg z_*$ is referred as dynamically completely rough wall.

Close to the smooth wall at $h_0 \ll z \ll z_*$, the viscous stress will be considerably greater that the Reynolds stress ($\nu \partial \bar{u} / \partial z \gg |u'u''|)$, and this layer is called the viscous sublayer. In accordance with (1.13), within this sublayer we may assume that

$$\nu \frac{\partial \bar{u}}{\partial z} = u_*^2 = \text{const}$$  \hspace{1cm} (1.17)

and therefore

$$\bar{u}(z) = \frac{u_*^2}{\nu} z$$  \hspace{1cm} (1.18)
That is, within the viscous sublayer the mean velocity profile is linear.
At greater distances from the wall when \( z \gg \max(z_*, h_0) \), the turbulent stress will be much
greater than the viscous stress, and we get from (1.13) and (1.11)
\[
-u'w' = K_M \frac{\partial \pi}{\partial z} = u_*^2 = \text{const} \tag{1.19}
\]
At \( z \gg \max(z_*, h_0) \) we may expect that the variation of mean velocity (i.e., \( \frac{\partial \pi(z)}{\partial z} \))
does not depend on the viscosity \( \nu \) and the protrusion height \( h_0 \) but is determined by \( u_* \) and \( z \) only. The
only possibility that follows from dimensional analysis is
\[
\frac{\partial \pi(z)}{\partial z} = \frac{1}{\kappa} \frac{u_*}{z} \tag{1.20}
\]
where \( \kappa \) is an universal constant usually called von Karman’s constant (from experiments, \( \kappa \approx 0.4 \)). From (1.20) and (1.19) we get for the wall turbulence expression for the eddy viscosity
coefficient
\[
K_M = \kappa u_* z \tag{1.21}
\]
The same expression will be also valid for the eddy diffusivity of heat: \( K_H \approx K_M \). Integrating
(1.20) we will get a well-known expression for velocity profile in the logarithmic boundary
layer
\[
\pi(z) = \frac{u_*}{\kappa} \ln \left( \frac{z}{z_0} \right) \tag{1.22}
\]
where \( z_0 \) is the roughens parameter. Note that formulae (1.20)-1(22) a valid in both cases of
smooth and rough wall provided that \( z \gg \max(z_*, h_0) \).
The roughens parameter characterizes the ability of a wall to resist the flow. If the wall is
dynamically smooth, the roughness parameter is expressed as
\[
z_0 = z_* \exp(-\kappa B) \approx \frac{1}{9} \frac{\nu}{u_*} \text{, } z_* \gg h_0 \tag{1.22'}
\]
where \( B \approx 5.5 \) is an universal constant. If the wall is completely rough the roughens parameter
is proportional to \( h_0 \)
\[
z_0 \approx \frac{h_0}{30} \tag{1.22''}
\]
The concept of logarithmic boundary layer is widely applied in physics of atmosphere and
ocean (in particular, in well-known POM model of ocean circulation [Mellor, 1993]).

**Drag Coefficient**

Usually one is able to measure a velocity of a turbulent flow at a level \( z = z_a \) only, and want to
get the value of the flow stress at the wall. It is possible if a drag coefficient for the wall is known
\[
\tau_0 = \rho u_*^2 = C_d \rho \pi^2(z_a) \tag{1.22'''}
\]
where \( C_d \) is the drag coefficient. Substituting (1.22) into (1.22′′′) we get

\[
C_d = \left[ \frac{1}{\kappa} \ln \left( \frac{z_o}{z_0} \right) \right]^{-2} \quad (1.22′′′)
\]

Therefore if we know the roughness parameter, we can calculate the drag coefficient.

Actually it is not a hard job to determine the roughness parameter if we deal with the truly rigid wall. But in many environmental flows, and the roughness parameter is not constant but can depend on a number of parameters like the flow velocity itself. For example, the roughness parameter of the sea surface will depend on parameters of surface waves (their direction, height, length, velocity etc.) which in their turn depend on the velocity and duration of the wind as well as the geometry of the basin. It makes the problem very complicated. Another example a wind blowing over a wheat field or a grass field when the roughness parameter of the “wall” will depend on the friction velocity \( u_* \) itself.

**Mixing-length concept**

An estimate of turbulent viscosity coefficient \( K_M \) can be obtained by analyzing the random motion of “fluid particles” in a turbulent shear flow \( \left( \frac{\partial u}{\partial z} \neq 0 \right) \) using the molecular exchange analogy. We can suppose schematically that vertical transport of momentum is because a particle moves up at a distance \( l \), and passes its momentum deficit to the ambient fluid, while an other particle moves down at the same distance \( l \) and passes its momentum excess to the environment, and so on. The length scale \( l \) is called the mixing length. Therefore, we can write for alongflow velocity fluctuation

\[
u^\prime = u_0^\prime - \frac{w^\prime}{|w|} \left[ \frac{\partial n}{\partial z} \right] \quad (1.23)
\]

where \( u_0^\prime \) is a “random” component of alongflow velocity fluctuation \( (u_0^\prime w^\prime = 0) \). (1.23) yields

\[
u^\prime w^\prime = -w^{1/2} \int \frac{\partial n}{\partial z} \Rightarrow K_M = \frac{w^{1/2}}{l} \quad (1.23^{'})
\]

where \( C \) is a constant of the order of unity. If measurements of turbulent velocity fluctuation are available, we can calculate \( w^{1/2} (z) \), and the mixing length \( l \) can be taken as the Eulerian integral scale of turbulence

\[
l(z) = \frac{1}{w^{1/2}(z)} \int_0^\infty w'(z + z_1)w'(z)dz_1 \quad (1.24)
\]

**Balance of kinetic energy of turbulence of horizontally uniform flat flow in vertically stratified fluid**

Let us define the instant and mean kinetic energy of turbulent fluctuations as
If we apply the method described just after (1.7)-(1.8) to a relatively simple case of horizontally uniform flow directed everywhere along the $x$ axis in a vertically stratified fluid, we can get the following equation for the mean kinetic energy of turbulence (the turbulent energy, for brevity)

$$
\frac{\partial b}{\partial t} = -\frac{\partial}{\partial z}\left(b' + \frac{p'}{\rho}\right)w' - u'w'\frac{\partial \overline{u}}{\partial z} - \frac{g}{\rho}\overline{\rho'w'} - \varepsilon
$$

(1.26)

where $\varepsilon$ is the rate of dissipation of kinetic energy of turbulence into heat, $z$ axis is directed upward. In the case of isotropic turbulence $\varepsilon$ can be expressed as

$$
\varepsilon = \frac{15}{2} \left(\frac{\partial u'}{\partial z}\right)^2
$$

(1.27)

(Full definition of $\varepsilon$ is $\varepsilon = \frac{3}{2} \sum_{i,j} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)^2$.

Equation (1.26) is extremely important in studies of oceanic turbulence, and it is worth to comment the terms of this equation in more detail.

$\frac{\partial b}{\partial t}$ - the time derivative of the turbulent energy (no comments);

$-\frac{\partial}{\partial z}\left(b' + \frac{p'}{\rho}\right)w'$ - the term describing turbulent diffusion of turbulent energy (like turbulent diffusion of momentum described by the Reynolds stress $u'w'$) and an effect of pressure fluctuation which cannot be explained in terms of clear physical sense;

$-u'w'\frac{\partial \overline{u}}{\partial z}$ - the generation of turbulence by the mean flow. This term is usually positive, because in accordance with (1.11) it may be expressed as $-u'w'\frac{\partial \overline{u}}{\partial z} = K_M \left(\frac{\partial \overline{u}}{\partial z}\right)^2 \geq 0$ if $K_M > 0$. It means that the mean flow feeds the turbulence with the energy;

$-\frac{g}{\rho}\overline{\rho'w'}$ - the sink/supply of turbulent energy due to the change of potential energy of stratification. If the density flux is directed upward ($\overline{\rho'w'} > 0$), the turbulence works against buoyancy force, and the turbulence spends its energy to mix a stratified layer i.e., $-\frac{g}{\rho}\overline{\rho'w'} < 0$. In this case the buoyancy flux can be parameterized in usual manner using the concept turbulent mixing coefficient:
If \( \rho' w' < 0 \) the buoyancy forces feed the turbulence with energy, i.e., \(- \frac{g}{\rho} \rho' w' > 0\). This case is called the convection and it cannot be parameterized properly by the concept of turbulent mixing coefficient (otherwise we would get to negative \( K' \));

\(-\varepsilon\) - the sink of turbulent energy to the heat. This term is always negative.

To evaluate the relative role buoyancy in generation of turbulent energy by comparison with the dynamic factors (e.g., the transfer of energy from the mean motion) let us introduce a nondimensional parameter

\[
R_f = - \frac{\frac{g}{\rho} \rho' w'}{u' w' \frac{\partial u}{\partial z}}
\]

(1.29)

called the flux Richardson number. It is clear that \( R_f < 0 \) for the case of convection, and \( R_f > 0 \) for usual case of hydrostatically stable stratification. With (1.29), equation (1.26) can be rewritten as

\[
\frac{\partial h}{\partial t} = - \frac{\partial}{\partial z} \left( b' + \frac{p'}{\rho} \right) w' - u' w' \frac{\partial u}{\partial z} (1 - R_f) - \varepsilon
\]

(1.30)

Since \( \varepsilon > 0 \), \(-u' w' \frac{\partial u}{\partial z} > 0\), it is clear from (1.30) that stationary (undumped), uniform in z direction turbulence in stratified fluid is possible only if

\[
R_f < 1
\]

(1.31)

If we express momentum and density fluxes in (1.29) as

\[
\frac{u' w'}{\rho} = -K_M \frac{\partial u}{\partial z}, \quad \rho' w' = -K_H \frac{\partial \rho}{\partial z} = \frac{\rho}{g} K_H N^2
\]

(1.32)

where

\[
N^2 = - \frac{g}{\rho} \frac{\partial \rho}{\partial z}
\]

(1.33)

is squared Vaisala-Brant frequency, the criterion (1.31) can be rewritten as

\[
R_i < \frac{K_M}{K_H}
\]

(1.34)

where

\[
R_i = \frac{N^2}{\left( \frac{\partial u}{\partial z} \right)^2}
\]

(1.35)
is the ordinary Richardson number. Note, $\frac{K_M}{K_H}$ being usually equal or greater than unity can
be interpreted as the turbulent Prandtl number. The Richardson number $Ri$ plays an important
role in dynamics of flows in stratified fluid, especially in instability problems.

**Kolmogorov theory of locally isotropic turbulence**

Let us consider the simplest case of stationary turbulence in a horizontally uniform shear flow
with vertically uniform shear in an incompressible uniform (nonstratified) fluid. In
accordance with (1.30), the turbulent energy balance will be reduced to two terms describing
generation of turbulent energy by the mean flow, and the dissipation of turbulent energy into
the heat

$$0 = -u'w' \frac{\partial \overline{\Pi}}{\partial z} - \varepsilon$$  \hspace{1cm} (1.36)

Suppose that the mean flow has a velocity $U$ and length scale $L$, and $Ri = UL/\nu >>>1$. The
mean flow will feed large turbulent eddies with energy. Velocity and length scales of these
large eddies is expected to be close to respective scales of the mean flow, so the rate of energy
supply to turbulence may be estimated as

$$-u'w' \frac{\partial \overline{\Pi}}{\partial z} \approx \frac{U^3}{L}$$  \hspace{1cm} (1.37)

where $U$ and $L$ are external scales of velocity and length for the turbulence. This energy
comes to turbulent eddies of the length scale $L$, and these eddies are anisotropic because they
“remember” their origin from a mean flow of some direction. Than, due to multiple stretching
(see the “family tree” shown in Fig.2) the energy is transferred from the large eddies to
smaller and smaller eddies, and these smaller eddies becomes isotropic because they do no
longer remember how the mean flow was directed. Finally, the smallest eddies which have
velocity and length scales $U_S$ and $L_S << L$, so $Re_S = U_SL_S/\nu \approx 1$ will pass their energy to the
heat with the rate of dissipation equal to the energy supply from the mean flow at large length
scales

$$\varepsilon = \frac{U^3}{L}$$  \hspace{1cm} (1.38)

The isotropy of small-scale eddies is called local isotropy. Because locally isotropic eddies do
not “remember” their origin, Kolmogorov, a great Russian mathematician, suggested that
parameters of these eddies do not depend on external scales of turbulence $U$ and $L$ separately
but on the energy supply $\frac{U^3}{L} = \varepsilon$. Because the smallest eddies is sure to depend on viscosity $\nu$,
Kolmogorov concluded that the spectral density of kinematic energy of locally isotropic
eddies $E(k)$, where $k$ is the wave number, $E(k)dk$ is the energy of eddies from wavenumber
range $(k, k+dk)$ will depend on $k$, $\nu$, $\varepsilon$. Therefore, the dimensional analysis yields the
following functional form for energy spectrum
\[ \frac{E(k)}{v^{5/4} \varepsilon^{1/4}} = \frac{E(k)}{v^2 \eta} = f(k \eta), \quad k \gg 1/ L \]  

(1.39)

where \( \eta = (v^3 / \varepsilon)^{1/4} \) is the Kolmogorov microscale (the length scale for the smallest turbulent eddies) and \( v = (v \varepsilon)^{1/4} \) is the Kolmogorov velocity (the velocity scale for the smallest turbulent eddies). The range of wavenumbers for which (1.39) is valid is called the equilibrium range.

If locally isotropic eddies is large enough with respect to the Kolmogorov microscale \( \eta \) and small enough with respect to the external length scale of turbulence \( L \), their spectrum will no longer depend on viscosity \( \eta \) but \( k \) and \( \varepsilon \) only. The dimensional analysis yields

\[ E(k) = \alpha \varepsilon^{2/3} k^{-5/3}, \quad \frac{1}{\eta} \gg k \gg \frac{1}{L} \]  

(1.40)

where \( \alpha \approx 1.5 \) is an universal constant. The range of wavenumbers for which (1.40) is valid is called the inertial subrange.

**An example of the second moment closure**

The most of science people are not interested in the structure of turbulence itself but the effect of turbulence on the mean flow and the heat/mass/etc. transport. This problem is called the parameterization of turbulence. By now, the most advanced, widely used method of parameterization of turbulence is based on a second moment closure schemes. Let us consider one applied in the POM model (Mellor, 1993)

The Reynolds stress is written is a Boussinesq manner

\[ (u'w', v'w') = -K_M \left( \frac{\partial \Pi}{\partial z}, \frac{\partial \Pi}{\partial z} \right), \quad w'T' = -K_H \frac{\partial T}{\partial z} \]  

(1.41)

Vertical diffusivities for momentum \((K_M)\) and scalars \((K_H)\) are defined according to

\[ K_M = q l S_M \quad K_H = q l S_H \]  

(1.42)

where \( q = b^{1/2} \) is the root square of the turbulent energy, \( l \) is the length scale of the turbulence, and \( S_M = S_M(G_H), S_H = S_H(G_H) \) are some empirical (taken from experiments) functions of the turbulent Richardson number

\[ G_H = \frac{N^2}{q^2 / l^2} \]  

(1.43)

where \( N^2 = -\frac{g}{\rho_0} \frac{\partial \Pi}{\partial z} \) is the Vaisala-Brant frequency. Expressions (1.42) were first suggested by Kolmogorov (for nonstratified media, without \( S_M(G_H) \) and \( S_H(G_H) \) inclusion).

To close the system of equations, the following equations for \( q^2 \) and \( q^2 l \) are suggested
\[
\frac{\partial q^2}{\partial t} + \frac{\partial \bar{u} q^2}{\partial x} + \frac{\partial \bar{v} q^2}{\partial y} + \frac{\partial \bar{w} q^2}{\partial z} = \frac{\partial}{\partial z} \left( K_q \frac{\partial q^2}{\partial z} \right) \\
+ 2K_M \left[ \left( \frac{\partial \bar{u}}{\partial z} \right)^2 + \left( \frac{\partial \bar{v}}{\partial z} \right)^2 \right] + \frac{2g}{\rho_0} K_H \frac{\partial \bar{\rho}}{\partial z} - \frac{2q^3}{B_1} l + F_{q^2}
\]
(1.44)

\[
\frac{\partial q^2 l}{\partial t} + \frac{\partial \bar{u} q^2 l}{\partial x} + \frac{\partial \bar{v} q^2 l}{\partial y} + \frac{\partial \bar{w} q^2 l}{\partial z} = \frac{\partial}{\partial z} \left( K_q \frac{\partial q^2 l}{\partial z} \right) \\
+ E_1 l \left[ K_M \left( \frac{\partial \bar{u}}{\partial z} \right)^2 + \left( \frac{\partial \bar{v}}{\partial z} \right)^2 \right] + E_3 \frac{g}{\rho_0} K_H \frac{\partial \bar{\rho}}{\partial z} \right) - \frac{2q^3}{B_1} \tilde{W} + F_{q^2 l}
\]
(1.45)

where \( K_q = q l S_q (G_H) \) is diffusivity for the turbulent energy, \( B_1, E_1, E_3 \) are some empirical constants, \( \tilde{W} \) is so-called wall proximity function (\( \tilde{W} = 1 \) in the fluid interior and \( \tilde{W} \to 0 \) when \( z \to H \) or \( z \to \eta \) (\( H \) and \( \eta \) are the sea depth and free surface elevation)), \( F_\phi \) is the horizontal flux function for a quantity \( \phi \)

\[
F_\phi = \frac{\partial}{\partial x} (A_H \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y} (A_H \frac{\partial \phi}{\partial y})
\]
(1.46)

where \( A_H \) is the horizontal diffusivity. Horizontal diffusivity is given from Smagorinsky formula

\[
(A_M, A_H) = (C_M, C_H) \Delta x \Delta y \left[ \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \right)^2 \right]^{1/2}
\]
(1.47)

where \( C_M, C_H \) are some empirical constants of the order of 0.1, \( \Delta x, \Delta y \) are the grid steps of a numerical scheme which is applied to solve the problem. The general way to obtain equations (1.44) and (1.45) was shown earlier (see text just after (1.7)-(1.8)). It includes a lot of hypotheses and simplifications. (To get an equation for \( q^2 l \), it can be formulated as \( q^2 l \sim q^5 / \epsilon \), and then the general way may be applied.)